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Arbitrage and Optimal Portfolio Choice with Financial Constraints

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Editorial

In this paper Helmut Elsinger and Martin Summer analyze the pricing of risky income streams in a world with competitive security markets and portfolio constraints. The authors investigate how one can transfer concepts and pricing techniques from a world without frictions to such a more realistic situation. Basically two new aspects arise: The no arbitrage condition has to be replaced by a weaker concept, which is called no unlimited arbitrage. Furthermore an appropriate technique is required for deriving from this concept a pricing theory for contingent claims. The authors show how to achieve this task in a simple way, which is applicable to many relevant constraint situations.

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Arbitrage and Optimal Portfolio Choice with Financial Constraints.*

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Abstract

We analyze the pricing of risky income streams in a world with competitive security markets where investors are constrained by restrictions on possible portfolio holdings. We investigate how we can transfer concepts and pricing techniques from a world without frictions to such a more realistic situation. We show that basically two new aspects arise: First, portfolio constraints can lead to situations where not all arbitrage opportunities are necessarily eliminated. For a world with portfolio constraints the concept of no arbitrage has to be replaced by a weaker concept which we call no unlimited arbitrage. Second, though we can characterize prices which allow no unlimited arbitrage by the existence of certain state prices as in the unconstrained case, additional computational work is needed for deriving from this fact a pricing theory for contingent claims. We propose a technique which can achieve this task and which renders itself for a computationally simple implementation for many constraint situations which are of practical interest. The power of no arbitrage techniques is preserved in the sense that no specific assumptions about utility functions of investors have to be made. We relate our analysis to the optimal decision problem of an investor and show the various relations between the properties of an optimal solution to this problem and the arbitrage-free values of risky income streams. This opens a unified view on the different approaches to asset pricing under portfolio constraints used in the literature and conveys their common underlying logic.

Keywords: Arbitrage, Portfolio Constraints, Asset Pricing
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1 Introduction

Pricing securities and risky income streams by *no arbitrage* arguments has become the cornerstone of modern asset pricing theory. No-arbitrage arguments have also been an impressive practical success. The valuation techniques derived from them have become the daily tools and workhorses of thousands of practitioners and financial engineers worldwide. The idea of no arbitrage is simple. It requires that correctly priced securities should make it impossible to achieve by financial transactions a consumption bundle at zero costs that increases some investor's utility. This idea ultimately relies on an equilibrium argument and has powerful implications for asset pricing formulas. A great deal of this power comes from the fact that the question whether or not security prices do allow arbitrage, can be inferred from observable data: the prices of actively traded securities and their payoff structure. We do not have to know the entire equilibrium. Moreover once the "correct" security prices have been found, the price of any risky income stream which can be generated by combinations of these securities is determined. Thus security pricing by no arbitrage leads to a general valuation technique for arbitrary contingent claims, which can be generated from securities traded on financial markets.

Yet the formulas are derived under highly idealized conditions. Among them, *perfect competition* and *frictionless security trading* are the two most important ones. Evidence as well as practical experience suggests that the assumption of price taking behavior is to a large extent fairly appropriate for financial markets for standard securities, such as options, futures, stocks and bonds. The assumption of frictionless trading - however - is surely inappropriate. Margin requirements, short selling restrictions, borrowing constraints and collateral requirements belong to the basic facts of (financial) life, even for the most competitive financial markets.

In this paper we ask *whether* and *how* we can transfer the *power and the simplicity* of pricing a risky income stream by no arbitrage arguments to a world where such constraints bind investors in their portfolio decisions. The answer we get is that this transfer is indeed possible but we have to introduce a new concept and we have to do some extra computational work. First of all it turns out that once portfolio constraints are taken into account the requirement that financial markets admit *no* arbitrage is too restrictive. We argue that the appropriate criterion we have to use in a world with frictions, is a concept which we call *no unlimited arbitrage*. Constraints can lead to situations where not *all* arbitrage opportunities are eliminated in equilibrium because the constraints prevent investors to fully take advantage of them. In parallel to the frictionless world we can *characterize* the requirement that financial markets admit no unlimited arbitrage by the existence of certain state prices. Unfortunately, contrary to the unconstrained world we are not ready for pricing income streams after we have obtained such a characterization of no unlimited arbitrage. We show that for any income stream that can be replicated we have to find among all the candidate state prices, the "correct" ones for a particular income stream. We propose a computationally simple procedure which is able to accomplish this task. It can be set up from the basic data of security prices, financial contracts with their payoff structure and the relevant constraints on feasible portfolio positions. We are thus able to present a pricing theory for arbitrary contingent claims that can be replicated by existing securities under constraints without requiring any particular knowledge of investor utility functions beyond some general assumptions on the structure of preferences.

Since it is our aim to analyze and clarify some of the conceptual questions that arise in transferring arguments in the spirit of no-arbitrage to a framework where investors are constrained in their potential portfolio holdings, we have decided to use a framework, which has the minimal structure that is able to address the issues in a meaningful way. The reader initiated to modern asset pricing theory and security pricing might thus perhaps miss the rich stochastic structure which has become a trade mark of this literature. We present our arguments in a framework that is stripped to the bare essentials to convey the basic logic of pricing contingent claims under constraints. Our results do however not depend on the simplified framework and can easily be generalized to richer setups.

The paper is organized as follows: Since at first sight all the different contributions to the pricing problem under constraints seem to offer their own (idiosyncratic) approach we have decided to start in section 2 with a discussion of the literature to put the papers including our own contribution into perspective. Section 3 gives an exposition of the model and introduces the formal description of constraints along with some examples. Section 4 describes the feasible income transfers in financial markets with portfolio constraints. Section 5 characterizes no unlimited arbitrage in terms of state prices. Section 6 demonstrates how this characterization can be used to price an arbitrary contingent claim by no unlimited arbitrage. Section 7 contains results which show the connections between the no arbitrage pricing approach and optimal investor decision problems. The final section 8 concludes. All proofs are in the appendix.

2 Related Research

We do not claim to be the first authors treating security pricing in the presence of portfolio constraints. In fact there is a growing literature on this topic building on a stock of seminal papers. To our best knowledge our paper is the first to propose the concept of no unlimited arbitrage as an appropriate tool for analyzing security markets with constraints and to derive suitable valuation techniques from this concept.[1] Our aim is to develop a framework in which conceptual issues can be discussed in a transparent way and which is capable to bring different approaches in the literature into a unified perspective.

In the following we give an overview on the recent literature on portfolio constraints which is most closely related to the ideas discussed in our paper. We suggest to classify the papers according to two broad categories. The literature in the first category approaches the valuation problem with constraints by extending a classical paper by Harrison and Kreps [21] (see also Kreps [26]) which discusses the case of an unconstrained financial market. The primitives by which the problem is approached there is an abstract linear space of *net-trades* together with a linear pricing function defined on this space. These two objects reflect in an abstract way frictionless trading of arbitrary risky income streams (the linear space property) and perfect competition (the linearity of the pricing functional). It is assumed that the economy is populated by agents with preferences over net trades about which *some* general properties are known. Among these, monotonicity ("more is better") is the most important one. In this context the question is asked: When can the pricing functional together with the feasible net trades be part of an economic equilibrium, if agents are known to have these general properties? (see Kreps [26, p. 20]) The answer to this question is then given by a characterization of a no arbitrage requirement via the existence of certain *state prices*. Thus the general idea is to approach the valuation problem without postulating a specific structure on agents' preferences besides of some general properties. This general idea is then extended to a world with financial constraints. The literature in the

[1] However Charupat and Prisman [7] in a critical note on a paper by Chen [8] have pointed out conceptual problems arising by naively transferring the no arbitrage conditions from a frictionless world to a world with constraints.

second category approaches the valuation problem by building on the analysis of solutions to an optimization problem of a representative investor, who can put his wealth into a riskless bank account and a set of risky securities the prices of which follow some stochastic process. The valuation question in this framework is answered by pricing any contingent claim using the utility gradient of the representative investor. Thus the general idea in this approach is to postulate a specific utility function and a specific stochastic model of security prices to add portfolio constraints and analyze the value of some given claim by solving the representative investor's utility maximization problem.

Our approach is in the spirit of the first category. Important papers in this literature are He and Pearson [20], Jouini and Kallal [24], [25], Huang [22]. Jouini and Kallal formulate their model by restricting the framework of Harrison and Kreps [21]. They take a *convex cone*[2] of net trades (instead of a linear space) and a *sublinear pricing function* (instead of a linear one) for contingent claims as a primitive.[3] Using these primitives, no arbitrage is characterized when some general properties on investors' preferences are assumed. Our model contains the result of characterizing no arbitrage, when net trades are constrained to be in a cone as a special case. Contrary to Jouini and Kallal, we take some effort to model *in detail* the role of security prices and financial markets for the pricing of arbitrary contingent claims. We achieve this by working in a slightly less abstract framework clearly distinguishing financial markets, the prices of securities and arbitrary contingent claims which can be generated by these securities under constraints as separate objects. We are thus able to make fully transparent under which conditions linear security prices (competition) and portfolio constraints (frictions) interact to actually *imply* a sublinear pricing function for arbitrary contingent claims. Huang (1998) conducts a similar analysis to ours in an infinite horizon event tree setting for the special case of constraint sets which are cones. Our paper discusses a more general class of constraints, because there are practically important situations for which this is indeed required. Furthermore, contrary to Huang, we discuss conceptual issues at some length and analyze the relation between no arbitrage, no unlimited arbitrage and optimal portfolio decision problems. Due to this aspect of our paper, we are able to also contribute to issues discussed in the literature on financial innovation, as documented by the papers of Allen and Gale [1], Chen [8] and Charupat and Prisman [7]. The paper of He and Pearson [20] in contrast to ours considers a smaller class of constraints. They give a characterization of arbitrage free prices under constraints for this special case but when they make use of the characterization to value an arbitrary contingent claim, they have to use a utility function. Resort to a utility function can be avoided in our approach.

Seminal papers in the second category are Cvitanić and Karatzas [11],[12]. These papers consider general convex constraint sets and have inspired further research, most notably Cuoco [9], Munk [29], [30], Tepla [35] and Detemple and Murthy [16]. Cvitanić and Karatzas have developed a technique which exploits duality theory in a skillful way to get arbitrage-free prices for contingent claims in a representative investor framework with portfolio constraints, where security prices follow a Brownian motion. Though we do not use this approach our discussion of optimal investor decisions and arbitrage free prices under constraints clearly conveys how the Cvitanić and Karatzas approach is related to the literature of category one. As a by-product we

[2] A non empty subset C of a real vector space V is called a *convex cone*, if $x \in C$, $\lambda \geqslant 0 \Rightarrow \lambda x \in C$, $\forall x, y \in C$: $x + y \in C$ (see Luenberger [27, p.18]).

[3] A real valued function f defined on a real vector space V is said to be sublinear on V if $f(x+y) \leq f(x)+f(y)$ for all $x, y \in V$ and $f(\lambda x) = \lambda f(x)$ for all $\lambda \geqslant 0$ and $x \in V$. As an example, consider any norm on V. By definition the norm is a real valued function, which is positive, homogeneous and fulfills the triangle inequality, hence is a sublinear function (see Luenberger [27, p. 110]).

demonstrate under which conditions their approach could be employed without making reference to a specific representative investor optimization problem (thus to a particular utility function).

In the context of the literature, our paper has the following contributions: First, we demonstrate in an elementary way how ideas of asset pricing by no arbitrage can be transferred to a world with portfolio constraints. Second, we derive a *practical and computationally simple technique* to value arbitrary (redundant) contingent claims under constraints using only basic information about traded securities, their prices and the constraint situation. Third, we show in a transparent way how competitive security prices and trading frictions interact to restrict the valuation of arbitrary contingent claims, thus highlighting the role played by (competitive) financial markets. Fourth, we *unify the different approaches* by conveying the underlying logic of the arguments and show how the papers which we have classified into two categories, are related.

3 The Finance Model With Portfolio Constraints

Consider the standard general equilibrium, finance model in its simplest version. There are two dates $t = 0, 1$ and a finite set $\mathcal{S} = \{1, ..., S\}$ of states of the world at date 1, describing uncertainty. There is a finite set $\mathcal{I} = \{1, ..., I\}$ of investors who wish to exchange a (numèraire) good, which we could think of as income. In order to do so they can competitively trade a finite set $\mathcal{J} = \{1, ..., J\}$ of financial contracts in quantities z at prices q at date 0. Financial contracts are promises to some payoff of the good in the different states at date 1 and are represented by a $S \times J$ matrix A.[4] Investors are characterized by a continuous, strictly quasi-concave, and strictly monotone utility function $u^i : \mathcal{R}^{S+1} \to \mathcal{R}$ and a vector $\omega^i \in \mathcal{R}^{S+1}_{++}$ of initial endowments of the good.

For the formal description of constraints, we assume that each investor $i \in \mathcal{I}$ can choose his *portfolios* z consisting of positions in the J contracts traded on the market not from $\mathcal{R}^J$, as it is usually assumed, but only from a closed, convex set $Z \subset \mathcal{R}^J$. To be precise we require

Assumption (CON): *Each Investor $i \in \mathcal{I}$ may choose his portfolio z^i from a closed, convex set $Z \subset \mathcal{R}^J$, which is non-trivial, i.e. $Z \neq \{0\}$ and which contains* 0.

Assumption (CON) allows to describe a fairly large class of practically important restrictions on portfolio holdings. To see this, let us consider some examples.

Example 1 ***Margin Requirements:*** *Margin requirements are common practice in security trading. In particular in derivative trading investors are required to keep margin accounts, which represent a performance bond. Margins are set by regulators, clearing houses and intermediaries. Many of the common margin requirements can be described by (CON). The particular form will depend on the specific margin requirement considered. An example of a margin requirement is for instance that security positions can only be chosen from the set*

$$Z = \left\{ z \in \mathcal{R}^J \mid q_j z_j \geqslant -m_j q z \text{ for } m_j \in \mathcal{R}_{++},\ j \in \mathcal{J} \right\}.$$

[4] We adopt the convention that all entries in A are non negative. In some parts of the finance literature such securities are called limited liability assets. Since we don't include into the model problems of bankruptcy and default, following the mainstream of the literature on asset pricing, this assumption can be made without loss of generality. The results do not depend on this assumption.

Thus the ability of investors to short sell certain securities is limited by the requirement to maintain an income margin, which is a (linear) function of their creditworthiness. Note that this example refers to margins in derivatives markets. A slightly different issue are margins required in equity trading. There the margin has the function of a down payment for the purchase of an equity and is de facto like a loan. When we talk of margin requirements we mean margin accounts with a performance bond function as it is common for instance in futures trading.

Example 2 ***Collateral Requirements:*** *Some securities traded on competitive financial markets can be used as debt instruments and have to be secured by an asset or a pool of assets, which are often other securities. Examples are collateralized swap contracts, collateralized mortgage obligations, collateralized depository receipts or collateralized bond obligations. One way to describe such constraints by (CON) can for instance be as follows: Let us divide a portfolio* $z \in \mathcal{R}^J$ *into assets and liabilities, depending on whether* $z_j > 0$ *or* $z_j < 0$. *Denote assets by* $z^+ = (\max[0, z_j])_{j=1}^J$ *and liabilities by* $z^- = (\min[0, z_j])_{j=1}^J$. *The requirement that liabilities are partially collateralized by assets can the be written in terms of the set* Z *as*

$$Z = \left\{ z \in \mathcal{R}^J \mid -qz^- \leq \theta q z^+ ,\, \theta \in [0,1] \right\}.$$

Example 3 ***Portfolio Mix Constraints or Target Ratios:*** *Constraints on the mix of a portfolio or target ratios for specific assets are common in security trading. These constraints can come from various sources, for instance regulations or corporate financial policies. Whenever we have a situation where constraints of this sort occur, we can use (CON) to describe it. In this cases the set* Z *can be described as*

$$Z = \left\{ z \in \mathcal{R}^J \mid \alpha q_j z_j \leq q_k z_k \leq \beta q_j z_j \text{ with } k, j \in \mathcal{J},\, 0 \leq \alpha \leq \beta \right\}.$$

Huang [22] has modelled debt to equity ratios in this way. These constraints require investors to keep the ratio of asset k *and* j *in a certain range determined by the bounds* α *and* β.

Example 4 ***Bid-Ask Spreads and Taxes:*** *Assumption (CON) is also able to model trading frictions expressed by different bid and ask prices, as studied for instance by Jouini and Kallal [25]. This can be formalized by considering two financial contracts* A^j, A^k, $j, k \in \mathcal{J}$ *with an identical payoff structure (i.e.* $A^j = A^k$*) one of which can't be sold short (*$z^j \geqslant 0$*), whereas there is a buying constraint on the other one (*$z^k \leq 0$*). As an example think of a riskless bank account, which can be used for saving and borrowing. This can be modelled as two uncontingent income streams* $\mathbf{1} \in \mathcal{R}_+^S$ *for the savings and for the borrowing account. The savings account must not be sold short, whereas the borrowing account can only be held in negative amounts. The restriction de facto makes two different assets out of* A^j, A^k *which will be reflected in different prices. The difference between these prices can be interpreted as a bid-ask spread. By the same logic one could use (CON) to describe the effects of taxes as in Prisman [32] or in Dybvig and Ross [18]*

From a formal viewpoint in all these examples Z is a *convex cone*. This is the case almost exclusively dealt with in many papers on arbitrage and portfolio constraints. However to capture some important additional portfolio constraint situations, which are practically relevant, let us point out that our weaker requirement that Z is just a (closed and) convex subset of $\mathcal{R}^J$ is indeed necessary. To see this consider the following:

Example 5 ***Short Selling Limits and Buying Floors:*** *Many securities are restricted in the amount that can be sold short. Stocks can usually not be sold short in large amounts or only at a very high cost. Buying constraints can occur, when some legal restrictions prevent holding of particular securities above some given threshold prescribed by the regulation. Constraints of this nature can easily be described by (CON). Consider for example different short selling limits on securities* $i = 1, ..., k$ *and buying floors for securities* $j = k+1, ..., J$*, then*

$$Z = \left\{ z \in \mathcal{R}^J \mid z_j \geqslant l_j,\ z_i \leq u_i,\ l_i, u_i \in \mathcal{R} \text{ with } j = 1, ..., k,\ i = k+1, ..., J \right\}.$$

For $l \neq 0$ or $u \neq 0$ the constraints do not generate a cone but rather a translation of a cone. Let $p = (l, u)$ then $Z - p$ is a cone. Following Luenberger [27] we will call this a cone with vertex p. From a formal point of view these constraint sets are almost like the cones discussed in the major parts of the literature but not quite. We will see that the role played by the vertex p is not as innocuous as one might assume at first sight. Note that the constraint set need not have a linear structure. If feasible portfolio holdings are functions of risk measures like Value at Risk feasible portfolio holdings might for instance become functions of volatility parameters.

Example 6 ***Capital Adequacy:*** *Constraints of the sort described in the previous example have become of particular interest during the last years, where capital adequacy has dominated the regulatory debate about financial markets. Capital adequacy is a risk management concept which requires that the capital of a financial organization is sufficient to protect its counterparties and depositors from on- and off-balance sheet market risks, credit risk, etc. The European Union has recently implemented capital adequacy rules and they have become particularly important in portfolio insurance. Capital adequacy rules work like a minimal capital requirement (see Bardhan [4]). The requirement can be a function of risk measures like for instance Value at Risk (Jorion [23]).*

Example 7 ***Risk Based Capital Requirements:*** *Sometimes capital market regulations can lead to constraint situations, where the portfolio set* Z *is bounded. Cvitanić [10] gives as one particular example situations where feasible security holdings are limited in potential long and short positions. For instance the regulation of insurance companies sometimes prescribes so called risk based capital requirements. These requirements limit the amounts that can be invested into assets of a certain (default) risk class. Combined with short selling limits such constraints lead to a set* Z*, which can't be described by a cone or a cone translation.*

These two examples belongs to a class of constraints which are of considerable practical importance. However in these cases Z is not a cone but rather just a closed and convex set. Assumption (CON) allows to describe these cases.

This discussion demonstrates that the consideration of more general constraint sets than those which are usually dealt with in large parts of the literature is indeed *required* to cover important situations occurring in the practice of financial markets.

Let us finally note that assumption (CON) is used in different, essentially equivalent, versions. For instance, a seminal paper by Cvitanić and Karatzas [11] works with constraints on proportions of initial endowment ω^i invested in various available assets (see also the textbook by Pliska [31]). Some authors model direct constraints on dollar amounts that can be invested. All these approaches can easily be translated into each other. In our view the description chosen here allows a particularly transparent description of the relation between competitive financial markets, portfolio constraints, state prices and the implied contingent claim values.

Note that Z contains 0. This property of the portfolio constraints is natural because a reasonable model should always allow for not making any financial market transactions and just consuming the endowment, whatever the constraints may be.

As a formal object the financial market model is a tuple $\mathcal{E} = \{(u^i, \omega^i)_{i=1}^I, (A, Z)\}$.

4 Achievable Income Transfers Under Constraints

Investors achieve consumption indirectly via competitive security trading. Because of portfolio constraints, however, each investor is confined to a restricted (future) consumption profile depending on the constraint set Z. If we add to A as the first row the vector $-q$ to form a new matrix

$$T = \begin{bmatrix} -q \\ A \end{bmatrix} \tag{1}$$

we can write the *net income transfers* achievable for consumer i by holding a portfolio $z^i \in Z$ as

$$\tau^i = Tz^i, \qquad z^i \in Z. \tag{2}$$

Using (2) we can define the feasible income transfers induced by the financial market. Let us introduce the following definition:

Definition 1 *The set of feasible income transfers induced by* (T, Z) *is denoted by*

$$\mathcal{C} = \{\tau \in \mathcal{R}^{S+1} \mid \tau = Tz, \quad z \in Z\}.$$

Since Z is a closed, convex set, and T is a continuous linear transformation, the set of achievable income transfers will also be convex. Indeed we can assert:

Lemma 1 *The set* $\mathcal{C}$ *of feasible income transfers is a convex subset of* $\mathcal{R}^{S+1}$ *containing* 0.

Proof: Appendix. □

Unfortunately the set $\mathcal{C}$ does not necessarily inherit the closedness of Z.[5] We preclude such a situation by assumption.

Assumption (CONC): *Each Investor* $i \in \mathcal{I}$ *may choose his portfolio* z^i *from a closed, convex set* $Z \subset \mathcal{R}^J$*, which is non-trivial, i.e.* $Z \neq \{0\}$ *and which contains* 0. *Z, A and q are such that* $\mathcal{C}$ *is closed.*[6]

Since many practically important portfolio constraints can be formally described as cones or translations of cones, we want to know whether the set of feasible income transfers inherits this structure from Z.

[5] Sufficient conditions that $\mathcal{C}$ will inherit the closedness property from Z under the mapping T would for instance include the cases where Z is compact or polyhedral.

[6] An assumption similar to this is used in a different context by Ross [34].

Lemma 2 *If Z is a convex cone with vertex p, $\mathcal{C}$ is a convex cone with vertex Tp.*

Proof: Appendix. □

5 Arbitrage and Portfolio Constraints

A central idea of modern asset pricing theory is the explanation of the value of securities by analyzing security prices $q \in \mathcal{R}^J$ which allow *no arbitrage*. Essentially this requirement expresses the idea that in any *equilibrium*[7] it should not be possible, by trading securities, to achieve a consumption bundle at zero costs that increases some investor's utility (see Kreps [26]). The reason is that investors with monotone preferences would then wish to take an unlimited position in the arbitrage portfolio to generate an unlimited consumption profile. The ability to take arbitrary portfolio positions is therefore one essential building block of pricing arguments which invoke a no arbitrage condition.

Though ultimately the pricing of risky income streams by no arbitrage indirectly relies on an equilibrium argument, much of its power comes from the fact that the question whether or not security prices are arbitrage free (and can therefore be part of some equilibrium) can be inferred from the payoff structure of securities, the matrix A and the observed security prices q. There is no need to know the entire equilibrium. A famous theorem of finance (see for instance Duffie [17]) demonstrates for the standard finance model that the absence of arbitrage is equivalent to the existence of implicit (strictly) positive values of income in the different states of the world - the so called *state prices* - such that the value of any security is exactly equal to the value of the future income stream it provides under these state prices. It can therefore be checked from (q, A) alone whether or not there is an arbitrage possibility. On this limited information it is thus possible to find out which $q's$ are consistent with some equilibrium. Strictly positive state prices which make securities zero-profit investments, *characterize* the absence of arbitrage in the standard finance model.

Most of the models transferring this kind of argument to a world with constraints work with a generalization exactly along these lines. It can be formulated in analogy to the unconstrained case: The financial market (q, A, Z) allows no-arbitrage if there exists no $z \in Z$ with $Tz > 0$[8]. Written in a slightly less condensed form the definition requires that there does not exist a $z \in Z$ with $-qz \geqslant 0$ and $Az \geq 0$ where at least either the first or one of the other S inequalities is strict.

However, when we consider general situations of convex constraints as formalized by assumption (CON) and illustrated by the various examples we gave before we have to be careful since such a characterization might be too strong and we need a slightly *weaker* criterion. The problem which arises when we consider constraints in the class described by (CON) can perhaps most clearly be seen in a toy example where Z is a cone with vertex p. The basic message of the example is that portfolio constraints can lead to a situation where security prices can allow in principle financial transfers that imply *limited* arbitrage opportunities. This situation can nevertheless be consistent with *some* equilibrium, since constraints make it impossible for individuals to take any advantage of them.

[7] For a formal definition of an *equilibrium* for a model with financial markets with no portfolio constraints see for instance Magill and Quinzii [28, Definition 8.2.].

[8] The inequality $x > 0$ for a vector $x \in \mathcal{R}^n$ means that all components of the vector are nonnegative and not all of them are zero. We have $x > 0 \Leftrightarrow x \in \mathcal{R}^n_+$ and $x \neq 0$.

Example 8 *Consider a model with no uncertainty, so that $\mathcal{S} = \{1\}$. There are two investors $i = 1, 2$ with endowments $\omega^1 = (8, 1)$ and $\omega^2 = (2, 14)$ who have both identical preferences described by the utility function*

$$u^i(x_0^i, x_1^i) = \log(x_0^i) + \log(x_1^i)$$

The payoff matrix of financial contracts is given by $A = (1, 1)$ and the constraint set is given by $Z = [-2, \infty) \times (-\infty, 2]$. So there is a short selling limit on security one and a buying floor on security two. Now it is easy to check that the security prices $q^* = (1, 1/2)$ and the consumption and security demands $x^{1*} = (5, 5)$ and $x^{2*} = (5, 10)$, $z^{1*} = (2, 2)$ and $z^{2*} = (-2, -2)$ form an equilibrium for this economy because at these prices each investor has solved his utility maximization problem and in the market for the good and for securities supply and demand are balanced.[9] *The example can perhaps most clearly be seen by looking at Figure 1, which shows the equilibrium.*

Insert Figure 1 about here.

Figure 1: With portfolio constraints the financial markets need not be arbitrage free even in equilibrium.

What can we learn from this example? Had we required that there exists *no* $z \in Z$ with $Tz > 0$ to exclude all prices which can't possibly be part of any equilibrium, we would have obviously discarded the prices $q^* = (1, 1/2)$. However these prices - as we have just seen - are consistent with some equilibrium and should therefore not be ruled out. Exactly this would have occurred, however, by applying the criterion of *no arbitrage*. From the example we can see that it is *not* necessary for an equilibrium that there exists no $z \in Z$ such that $Tz > 0$. Obviously such $z's$ do exist and yet we have an equilibrium because constraints prevent the advantageous use of these opportunities by the agents of the economy.[10]

One can take a geometric viewpoint on the requirement of no arbitrage. It is equivalent to the condition that $\mathcal{C} \cap \mathcal{R}_+^{S+1} \backslash \{0\} = \emptyset$. This condition is not fulfilled in the equilibrium of the example because there $\mathcal{C}$ is a cone with vertex $\kappa = (1, 0)$. Therefore $\mathcal{C}$ can not have an empty intersection with $\mathcal{R}_+^{S+1} \backslash \{0\}$.

This example suggests the following *weaker* criterion for a world with portfolio constraints described by (CONC).

Definition 2 ***(NUA)*** *The financial market (q, A, Z) allows no unlimited arbitrage if there exists a vector $\kappa \in \mathcal{C}$ such that there is no $z \in Z$ with $Tz > \kappa$.*

In analogy to the unconstrained world we can *characterize* this requirement by the existence of certain state prices. We assert the following

[9] To calculate an equilibrium apply Definition 8.2. p. 69, in Magill and Quinzii [28], replacing the condition $z^i \in \mathcal{R}^J$ by the condition $z^i \in Z$.

[10] We have to mention that problems of equilibrium mispricing with features similar to those in our example have been already pointed out in the early literature on tax arbitrage.(Damon and Green [13], Ross [34]). In the context of portfolio constraints this problem has been described by Basak and Croitoru [3] and in an earlier paper by Charupat and Prisman [7]. These results - in particular the paper by Charupat and Prisman [7] - seem to have remained largely unnoticed in the literature on asset pricing with portfolio constraints.

Theorem 1 *(q, A, Z) admits no unlimited arbitrage if and only if there exists a vector $\bar{\pi} \in \mathcal{R}^{S+1}_{++}$ and a vector $\tau^* \in \mathcal{C}$ such that $\bar{\pi}\tau \leq \bar{\pi}\tau^*$, $\forall \tau \in \mathcal{C}$.*

Proof: Appendix. □

What does this theorem tell us about security prices that can be part of some equilibrium? To get an economic interpretation it is useful to write the inequality, which appears in the characterization of our theorem in a slightly less condensed form. In order to do so, let us normalize $\bar{\pi}$ by $\bar{\pi}_0$ so that we get

$$\left(\frac{1}{\bar{\pi}_0}\right)\bar{\pi} = (1, \bar{\pi}_{\mathbf{1}}).$$

Let $\tau^* = (\tau_0^*, \tau_1^*)$. Then the no unlimited arbitrage inequality can be written as

$$qz \geqslant \ \bar{\pi}_{\mathbf{1}}Az - (\tau_0^* + \ \bar{\pi}_{\mathbf{1}}\tau_1^*) \quad \forall z \in Z,\ \tau^* \in \mathcal{C} \tag{3}$$

In general in a situation with convex constraints of the portfolio holdings the present value of any traded security must be *larger or equal to the present value of the income stream under the state prices $\bar{\pi} = (1, \bar{\pi}_{\mathbf{1}})$ corrected by the present value of additional transfers provided by the limited arbitrage possibility induced by an appropriately chosen $\tau^* \in \mathcal{C}$.*

A problem with this characterization is obviously that typically there are many τ^* and $\bar{\pi} \in \mathcal{R}^{S+1}_{++}$ fulfilling the inequality $\bar{\pi}\tau \leq \bar{\pi}\tau^*$, $\forall \tau \in \mathcal{C}$. Moreover these $\bar{\pi} \in \mathcal{R}^{S+1}_{++}$ will differ depending on the τ^* that is chosen. Let $\Pi(\tau^*)$ denote the set of all $\bar{\pi} \in \mathcal{R}^{S+1}_{++}$, which fulfill the inequality $\bar{\pi}\tau \leq \bar{\pi}\tau^*$, $\forall \tau \in \mathcal{C}$ for a given feasible $\tau^* \in \mathcal{C}$. The set of *all* potential state prices Π, which are consistent with the requirement of no unlimited arbitrage, is characterized by the union $\cup_{\tau^* \in \mathcal{C}} \Pi(\tau^*)$, i.e. $\Pi = \cup_{\tau^* \in \mathcal{C}} \Pi(\tau^*)$. If worse comes to worst, this union might be rather too large to provide a reasonably sharp characterization. Moreover the whole construction looks quite bulky at first sight. These remarks can perhaps best be seen in the following figure, illustrating an example with so called "rectangular constraints" (Cvitanić 1997).

Insert Fig. 2 about here.

Figure2: Rectangular Constraints.

The examples have shown that many of the practically important constraint situations described by (CONC) are cones or translations of them. For this important class of constraints we are able to give a characterization of no unlimited arbitrage by using the vertex Tp of $\mathcal{C}$.

Corollary 1 *If Z is a cone with vertex p, the financial market (q, A, Z) admits no unlimited arbitrage if and only if there exists a vector $\bar{\pi} \in \mathcal{R}^{S+1}_{++}$ such that $\bar{\pi}\tau \leq \bar{\pi}Tp$, $\forall \tau \in \mathcal{C}$.*

Proof: Appendix. □

Thus for constraint sets which are cones with vertex p we know that all state prices which allow no unlimited arbitrage must lie somewhere in the set.

$$(\mathcal{C} - Tp)^{\ominus} := \left\{\pi \in \mathcal{R}^{S+1} |\ \pi\tau \leq \ \pi Tp,\ \forall \tau \in \mathcal{C}\right\} \tag{4}$$

which is just the negative conjugate cone of $\mathcal{C} - Tp$.[11] Thus in the case of cones or cone translations the set of all candidate state prices can be constructed by polarity from the knowledge of the set $\mathcal{C}$ of feasible income transfers induced by financial markets.

This important special case, gives us an opportunity to relate our results to other characterizations of no arbitrage in the presence of constraints, which have been suggested in the literature.

The case almost exclusively dealt with there are constraint sets, which are cones. In particular this case is investigated in the papers by He and Pearson [20], Jouini and Kallal [24], Jouini and Kallal [25] and Huang [22].

For cones the condition of no unlimited arbitrage reduces to the familiar requirement of *no* arbitrage, which is stated in the following

Corollary 2 *If Z is a cone, (q, A, Z) admits no unlimited arbitrage if and only if there exists a vector $\bar{\pi} \in \mathcal{R}^{S+1}_{++}$ such that $\bar{\pi}\tau \leq \ 0, \forall \tau \in \mathcal{C}$.*

Proof: This is a direct consequence of Theorem 1 and Corollary 1. □

By (4) we see that in this case we get a direct generalization of the results for a world without frictions. There the set of income transfers is a linear space and its polar cone is its orthogonal complement.

If we normalize $\bar{\pi}$ as above, the *no-arbitrage* inequality implies that security prices must fulfill the relation:

$$qz \geqslant \ \bar{\pi}_{\mathbf{1}} Az \quad \forall z \in Z \tag{5}$$

for a strictly positive $\bar{\pi}_{\mathbf{1}}$.

The economic interpretation of this inequality is that security prices are arbitrage free if and only if they are *larger or equal to the present value of the future income stream* provided by the corresponding securities traded on the financial markets. Potential discrepancies do not generate arbitrage opportunities because the constraints on the feasible portfolio positions prevent investors from taking advantage of these opportunities.

This is the result which has repeatedly been obtained in the literature on portfolio constraints independently by He and Pearson [20] and by Jouini and Kallal [24] and Huang [22]. Our discussion shows that in fact a constraint set which is a cone is needed to get (5) as an *appropriate* characterization of prices which can be part of some equilibrium. It can be seen as a special case of no unlimited arbitrage with a $\kappa = 0$.

Our inequality for cones is exactly the statement of Jouini and Kallal's Theorem (2.1) (see Jouini and Kallal [24]). Their theorem says that no arbitrage is equivalent to the existence of a positive linear functional (in our case π), with the property that its restriction to $\mathcal{C}$, lies below the contingent claim pricing functional. This is exactly what is expressed in condition (5). Though we don't know the pricing functional for an arbitrary income stream $y \in \mathcal{C}$ yet, we know that in any way it must fulfill inequality (5). We see that more generally this condition has to be modified and can be appropriately applied only for the case of constraints which are a cone.

[11] Let X be a vector space which is equipped with the inner product $\langle x, x^* \rangle$. Denote by X^* its dual space. Given a set $S \subset X$, the set $S^{\ominus} = \{x^* \in X^* : \ \langle x, x^* \rangle \leq 0 \text{ for all } x \in S\}$ is called the negative conjugate cone of S (see Luenberger [27, p. 157]).

6 Asset Pricing and Portfolio Constraints

6.1 Pricing Arbitrary Income Streams

Having characterized security prices allowing no unlimited arbitrage by the existence of certain state prices, how do we get a useful theory of *asset pricing* with financial constraints out of this characterization?

The problem we have is that infinitely many state prices fulfill the NUA condition, each one leading to a *different present value* for the income stream. Which one should we take to price an arbitrary claim in the set of feasible transfers?

Note that the situation here differs from a situation with incomplete markets. There we can also get an infinity of state prices π which characterize an arbitrage free security market (q, A). Yet this doesn't create a problem for asset pricing. Even if we have an infinity of state prices, with incomplete markets all these $\pi's$ have to agree on the subspace of income transfers induced by $\langle A \rangle$, the *marketed subspace* (see Magill and Quinzii [28]).

To see this remember that in a frictionless world no arbitrage is equivalent to the existence of a strictly positive vector of state prices π such that $q = \pi A$. In a complete market π is unique, in an incomplete market there will exist many positive π that fulfill the equality $q = \pi A$. Their common feature is that all of them assign the same present value to any $y \in \langle A \rangle$.

Mathematically this means that the *projections* of all $\pi's$, which fulfill the no arbitrage condition, onto the marketed subspace are equalized (see Magill and Quinzii [28, Corollary 12.6.]) whereas projections onto the orthogonal complement of the marketed subspace will depend on the chosen π. This is the reason why we are able to price income streams y which are replicable, i.e. $y \in \langle A \rangle$, while we are not able to price nonredundant income streams, i.e. $y \notin \langle A \rangle$. For given security prices q the price of any income stream $y \in \langle A \rangle$ is uniquely defined by

$$c_q(y) = \bar{\pi}_{\mathbf{1}} y = qz \qquad \text{for any} \quad z \in \mathcal{R}^J \text{ such that} \quad y = Az \tag{6}$$

(see Magill and Quinzii [28]). This relation lies at the heart of the famous pricing formulas of modern finance, like for instance the celebrated Black-Scholes Option pricing formula (Black and Scholes [6]). It says that the absence of arbitrage is equivalent to the requirement that there exist positive state prices, such that the value of an income stream, which can be "replicated" by the existing securities, must be equal to the value of its parts (under these state prices). If there are infinitely many state prices fulfilling the no arbitrage equation, each of them will assign the same present value to any replicable income stream (but different values to any $y \notin \langle A \rangle$).

Thus in a situation of incomplete markets, once we have characterized the absence of arbitrage by the existence of certain state prices, we are ready to price any contingent claim, that can be replicated.

In contrast, in a world with constraints things are not so simple since then there is in general no reason why the projections of the $\pi's$ fulfilling the NUA condition (3) onto the set of redundant income streams should be equalized. Hence after having found a characterization theorem for no unlimited arbitrage we are still not ready for pricing replicable income streams, since we have among all the candidates to find the one which assigns the "correct" value to a given (redundant) income stream y. Thus in a world with portfolio constraints, for each redundant income stream among all the potential state prices which characterize the (NUA)-condition, we have to find the ones which correctly prices this income stream. This is an *additional* problem, which arises in the presence of portfolio constraints. Though no unlimited arbitrage can be characterized by the existence of certain state prices, they are not directly useful for pricing arbitrary redundant income streams since any of them will in general produce a different present value for the given income stream.

The set of all feasible portfolios $z \in Z$ generates a subset $\mathcal{C}_1 \subset \mathcal{R}^S$ induced by (A, Z). $\mathcal{C}_1$ is the set of all possible date 1 income streams which can be generated by investing in the J basic securities, in the quantities available in Z. In analogy with the discussion in Magill and Quinzii [28] we call this set the *marketed subset*.

Definition 3 *The subset $\mathcal{C}_1 \subset \mathcal{R}^S$ generated by (A, Z) is called the marketed subset, where*

$$\mathcal{C}_1 = \{y \in \mathcal{R}^S \mid y = Az, \text{ and } z \in Z\}.$$

Our aim is to price a replicable income stream in the marketed subset and to keep to the spirit of no arbitrage pricing by just *implicitly* invoking the monotonicity assumption on preferences. We suggest an argument which recovers the correct state prices indirectly by duality. For a given income stream in the marketed subset, which can be replicated, monotonicity of preferences implies that among equivalent opportunities to hedge a given income stream in $\mathcal{C}_1$ by basic securities an investor will take the cheapest one. We will show below how this argument allows us to price any y which can be replicated with basic securities by the "correct" state prices, without making reference to a specific investor decision. We can price any feasible income stream *without having to know a specific utility function*.

A caveat is in order here. If we have a new asset with a payoff structure of an income stream which can be replicated under constraints, naively pricing this new asset with reference to replicating portfolios will be misleading if this new asset changes the marketed subset. This seems to be largely ignored in the literature. In this case we would also have to know which constraints for such an asset are relevant to decide whether it is redundant. Otherwise it is not possible to price it. This problem can not arise in an incomplete market. There the fact that an income stream y can be replicated implies that the introduction of an asset which pays y does not change the span of the marketed subspace.

So what do we mean if we talk of pricing arbitrary income streams? On the one hand we can assign a value to any income stream in the marketed subset. However without further qualification we can *not* conclude from this fact that this would be the price of an asset paying this income stream. This would require additional knowledge about relevant constraints for this asset. On the other hand we can use our theory to check whether existing assets are consistently priced relative to each other.[12]

6.2 Arbitrage Free Pricing of Income Streams under Constraints

In the light of the previous discussion, we define the cost of a date one income stream $y \in \mathcal{R}^S$ as the minimal cost that have to be incurred to hedge this income stream within the marketed

[12] Note that modern asset pricing theory which uses arguments based on arbitrage considerations is in all its variants only applicable to redundant claims. This is the main reason why a straightforward extension of modern asset pricing theory to price financial innovations has turned out to be a problem of considerable difficulty.

subset $\mathcal{C}_{\mathbf{1}}$. This might not be possible. Thus we can only price income streams, which can be replicated as a combination of basic securities, which generate $\mathcal{C}_{\mathbf{1}}$.

In general this problem is a non-linear programming problem (because Z can be any convex set), which reduces in the case where Z is a polyhedral to a linear programming problem. In fact most of the practical problems of interest can be formulated as linear programming problems. The following discussion applies duality theory to show the connection between the vector of no unlimited arbitrage state prices, the security prices q and the value of an income stream $y \in \mathcal{C}_{\mathbf{1}}$, defined as the equivalent value of its minimal replication costs. Assume $\overline{q}$ is a security price vector which allows no unlimited arbitrage. We define the price of an income stream $\overline{y} \in \mathcal{C}_{\mathbf{1}}$ by[13]

$$c_{\overline{q}}(y) = \min \overline{q} z \qquad \text{for any} \quad z \in Z \text{ such that} \quad \overline{y} \ \leq Az. \tag{7}$$

The minimum cost hedging problem can be stated as

$$\begin{array}{rc} & \min \overline{q} z \\ s.t. & z \in Z \\ & Az \ - \ \overline{y} \ \in \mathcal{R}_{+}^{S} \end{array} \tag{8}$$

Now for this problem we can assert the we the following

Theorem 2 *If the financial market (q, A, Z) admits no unlimited arbitrage and there exists a $\bar{z} \in Z$ such that $A\bar{z} \geq \overline{y}$, then there exists a solution to the optimal hedging problem (8), i.e. there exists a $z^* \in Z$ such that $c_{\overline{q}}(\overline{y}) = \overline{q} z^*$ and $Az^* \geq \overline{y}$.*

Proof: Appendix.□

Adding additional (weak) qualifications we can guarantee that the dual to this problem always has a solution. From this dual problem we can determine the "correct" state price vector to price by no unlimited arbitrage for a given income stream which is replicable under constraints. We have:

Theorem 3 *If the financial market $(\overline{q}, A, Z)$ admits no unlimited arbitrage and*
(a) either Z is polyhedral and there exists a $\overline{z} \in Z$ such that $A\ \overline{z} \geqslant \overline{y}$
or
(b) there exists a $\hat{z} \in Z$ such that $A\ \hat{z} > \overline{y}$,
the dual problem

$$\max_{\pi_{\mathbf{1}} \in \mathcal{R}_{+}^{S}} (\pi_{\mathbf{1}} \overline{y} - \ \sup_{z \in Z}(-\overline{q} z + \pi_{\mathbf{1}} A z))$$

has a solution and the optimal values of the primal and the dual problems coincide.

[13] The definition of the value of an income stream y as the minimum cost of super-replicating this income stream with a constrained portfolio is a well established approach in the literature. (See Bensaid, Lesne, Pagés, Scheinkmann [5], Edirisinghe, Naik, Uppal [19], Dermody and Rockafellar [14], [15]. Our aim is to show how this approach is related to the concept of no unlimited arbitrage analyzed in Theorem1.

Proof: Appendix.□

What does this theorem tell us? It says that for any income stream $\overline{y} \leq A\overline{z}$ for some $\overline{z} \in Z$ its present value implied by the solution to the minimal cost hedging problem *is the maximal present value of this income stream with respect to the (dual) state prices minus the value generated by the limited arbitrage opportunity.* This expresses the additional pricing element, which enters the valuation formula in the presence of constraints.

Why can we identify the value of the function $\sup_{z \in Z}(-qz + \pi_{\mathbf{1}}Az)$ (which is never smaller than 0) with the value of limited arbitrage opportunities? Remember that our characterization Theorem told us that no unlimited arbitrage is equivalent to the existence of a strictly positive vector of state prices $\overline{\pi} \in \mathcal{R}^{S+1}_{++}$ and some $\tau^* \in \mathcal{C}$ such that (after normalizing by the date zero state price) we get the formula:

$$qz \geqslant \ \overline{\pi}_{\mathbf{1}}Az - (\tau_0^* + \ \overline{\pi}_{\mathbf{1}}\tau_1^*) \quad \forall z \in Z, \text{ with } \tau^* \in \mathcal{C}$$

This inequality can be rewritten as

$$(\tau_0^* + \ \overline{\pi}_{\mathbf{1}}\tau_1^*) = \sup_{z \in Z}(-qz + \pi_{\mathbf{1}}Az) \qquad (9)$$

where the left hand side is just the present value of the consumption bundle $\tau^* = (\tau_0^*, \tau_1^*)$.[14]

From this discussion we can derive two useful corollaries for two practically important cases of constraints. Consider first the case where Z is a cone with vertex p.

Corollary 3 *If the financial market (q, A, Z) admits no unlimited arbitrage and Z is a convex cone with vertex p and there exists a $\overline{z} \in Z$ such that $A\overline{z} \geq \overline{y}$ the optimal hedging problem (8) has a solution. The dual problem can be stated as:*

$$\begin{array}{rl} & \max\ (\pi_{\mathbf{1}}y + (q - \pi_{\mathbf{1}}A)p) \\ s.t. & (1, \pi_{\mathbf{1}}) \in (\mathcal{C} - Tp)^{\ominus} \cap \mathcal{R}^{S+1}_{+} \end{array}$$

If either condition (a) or (b) of Theorem 3 hold, the dual problem is solvable.

Proof: Appendix. □

As a special case we get the following corollary for the cone.

Corollary 4 *If the financial market (q, A, Z) admits no unlimited arbitrage and Z is a convex cone and there exists a $\overline{z} \in Z$ such that $A\overline{z} \geq \overline{y}$ the optimal hedging problem (8) has a solution. The dual problem can be stated as:*

$$\begin{array}{rl} & \max\ \overline{\pi}_{\mathbf{1}}y \\ s.t. & (1, \overline{\pi}_{\mathbf{1}}) \in \mathcal{C}^{\ominus} \cap \mathcal{R}^{S+1}_{+} \end{array}$$

If either condition (a) or (b) of Theorem 3 hold, the dual problem is solvable.

[14] To relate equation (9) to the existing literature, in particular to Cvitanić and Karatzas ([11]), note that the right hand side of the equation is known as the *support function* of $-Z$, which we denote by $\sigma_{-Z}(\kappa)$, where $\kappa \equiv q - \pi_{\mathbf{1}}A$ (see Aubin [2]). Associated to the support function is the set of all $\kappa \in \mathcal{R}^J$ where $\sigma_{-Z}(\kappa)$ is finite. This set is a convex cone and is called the *barrier cone* $b(-Z)$ of $-Z$. If the dual problem has a solution $\sigma_{-Z}(\kappa)$ has to be finite and therefore $q - \pi_{\mathbf{1}}A$ has to be an element of $b(-Z)$.

Proof: Follows from the proof of Corollary 3. □

Therefore whenever the constraint set is a cone (or a translation of a cone), pricing by applying the principle of no unlimited arbitrage boils down to a quite straightforward procedure, which requires two things. Construct the set of all state prices by polarity. Then for an income stream, which is replicable, determine its value by solving the linear programing problem formulated in the corollary. This procedure has been discussed in Huang [22] and applied to the pricing of income streams in a model with an infinite time horizon and an event tree structure.

Another interesting conclusion can be drawn for both of these important cases. It turns out that *only* in the case of a cone the pricing functional which emerges from the solution of these problems has to be sublinear. Now we see which conditions are needed that competitive security markets with portfolio constraints actually *imply* that the pricing functional $c_{\overline{q}}(y) = \max_{\pi_{\mathbf{1}} \in \mathcal{R}^S_+} (\pi_{\mathbf{1}} y - \sigma_{-Z}(q - \pi_{\mathbf{1}} A))$ for arbitrary contingent claims is sublinear.

Corollary 5 *If Z is a cone then the pricing functional $c_{\overline{q}}(y)$ is sublinear.*

Proof:.Appendix. □

Note however that for arbitrary convex constraints the pricing functional is in general *not* sublinear. To see this consider the following

Example 9 *Consider again a case without uncertainty. The payoff matrix of financial securities is given by $A = (1, 1)$ and the constraint set is given by $Z = [-2, \infty) \times (-\infty, 2]$. The security prices are given by $q = (1, 1/2)$. Let $y = 2$. It is easy to see that the minimum hedging costs for y are 1. Now consider $y = 4$ which would cost 3. Therefore the costs of $y = 4$ are more than twice as much as for $y = 2$.*

Insert Figure 3 about here.

Figure 3: For general convex constraints the contingent claim pricing functional need not be sublinear.

From this discussion we see, that the sublinearity of the pricing functional is *implied* for constraints which are a cone but it is not implied in general.

From the perspective of asset pricing, we can claim that the pricing of redundant contingent claims in a world with well-behaved frictions, is (almost) as straightforward as in a frictionless world. Thus the assertion of Theorem 2 together with the corollaries for cones and translations for cones can be the basis for more realistic, and nevertheless practical, security pricing formulas.

7 Optimal Portfolio Decisions and Asset Prices

So far we have used an argument which has invoked the preferences of investors only indirectly, in the requirement that in any equilibrium there must be no unlimited arbitrage. We didn't need the utility function in any way for the pricing of income streams. We know that under the no unlimited arbitrage assumption, this can be done using the data (q, A, Z) and the monotonicity of the preferences only.

We now show that optimal decisions of investors and arbitrage free prices are intimately connected in the world with portfolio constraints in *almost* the same way as in the frictionless world. This discussion will also shed light on the unifying principles underlying competitive asset pricing under portfolio constraints.

The budget set of an investor $i \in \mathcal{I}$ in the finance model with assumption (CONC) is given by

$$B^i(q, \omega^i, A, Z) = \left\{ x^i \in \mathcal{R}_+^{S+1} \mid x^i - \omega^i \leq T z^i, \ z^i \in Z \right\}. \tag{10}$$

The investor's optimal decision problem is given by

$$\max \left\{ u^i(x^i) \mid x^i \in B^i(q, \omega^i, A, Z) \right\}. \tag{11}$$

The existence of a solution to this problem is equivalent to the absence of unlimited arbitrage in financial markets. Indeed we can assert:

Proposition 1 *Problem (11) has a solution if an only if there is no unlimited arbitrage.*

Proof: Appendix. □

The solution to the optimal decision problem of an investor is intimately related to prices which allow no unlimited arbitrage. We now show that the arbitrage free pricing vectors of a given income stream in the marketed subset must lie in the normal cone to the investor's budget set at this point. Before we do this we define

Definition 4 *The normal cone to the budget set B^i at x^{i*} is defined as*

$$N_{B^i}(x^{i*}) = \left\{ \pi \in \mathcal{R}^{S+1} \mid \pi(x^{i*} - x^i) \geqslant 0, \forall x^i \in B^i(q, \omega^i, A, Z) \right\}$$

Now introducing a differentiability assumption on the utility function to simplify exposition we claim:

Proposition 2 *Assume u^i is also differentiable. Thus $\nabla u^i(x^i) > 0 \ \forall x^i \in B^i(q, \omega^i, A, Z)$. If x^{i*} is a solution to (11) then $\nabla u^i(x^{i*}) \in N_{B^i}(x^{i*})$ and $\lambda \nabla u^i(x^{i*})$ is a state price vector for $\lambda > 0$.*

Proof: Appendix. □

Compare the statement of the above proposition to the case of a frictionless world. There the marketed subset is a linear space $\mathcal{C} = \langle T \rangle$ and the polar cone is equal to its orthogonal complement and thus $\mathcal{C}^\ominus = \langle T \rangle^\perp$. The normal cone at *every* consumption bundle in the boundary of the budget set coincides with $\langle T \rangle^\perp$ for any $\pi \in \mathcal{R}_{++}^S \cap \langle T \rangle^\perp$.[15] Thus the pricing of an income stream in this case becomes simpler, since we don't have to choose a particular $\pi \in \langle T \rangle^\perp$ for the income stream we wish to price. Any $\pi \in \langle T \rangle^\perp$ we wish to choose, will do the job. By contrast

[15] Note that because of monotonicity only boundary bundles will ever be demanded .

in a world with frictions the appropriate state price vector will depend on the particular income stream we want to price.

We have suggested to find the appropriate state price vector by solving the dual to the minimal cost hedging problem for a given income stream. From proposition 2 we see that if we price an arbitrary stream $y \in \mathcal{C}$ by calculating the minimum replication costs, we have in general no guarantee that there actually is an investor who values y that much. We know that every utility gradient is some state price vector and that every no unlimited arbitrage state price vector is among the feasible vectors in the dual of the minimal cost hedging problem. Still without using any further information in general the minimum replication costs will give an upper bound for the value of an income stream. However again for the important cases of cone constraints and translations of cones for every state price we get as solution to the minimal cost hedging problem, we will be able to find some investor fulfilling our assumptions, for whom this state price vector is a utility gradient.

The analysis of the optimal investment problem gives us an opportunity to connect our analysis, with the seminal approach by Cvitanić and Karatzas [11] to pricing contingent claims under constraints (see also the textbook by Pliska [31]).

These authors suggest basically the following procedure. Consider an *unconstrained* financial market economy $\mathcal{E} = \{u, \omega, A\}$. For given security prices q we construct an artificial price vector $p = q - \kappa \in \mathcal{R}^J$ where κ is taken from the barrier cone obtained from some portfolio constraint set which interests us. Thus we add some $\kappa \in b(-Z)$. In this unconstrained economy we consider a *family* of sets of feasible income transfers parametrized by κ. Corresponding to the matrix T we define for each $\kappa \in b(-Z)$ the matrix

$$T_\kappa = \begin{bmatrix} -q + \kappa \\ A \end{bmatrix}.$$

Thus

$$\mathcal{C}_\kappa = \langle T_\kappa \rangle.$$

The space $\langle T_\kappa \rangle$ is a linear subspace of $\mathcal{R}^{S+1}$. For this economy we have a family of the budget sets for the representative investor given by

$$B_\kappa(q - \kappa, \omega, A) = \left\{ x \in \mathcal{R}_+^{S+1} \mid x - \omega \leq T_\kappa z, \; z \in \mathcal{R}^J \right\} \tag{12}$$

The investor's optimal decision problem is given by

$$\max \{u(x) \mid x \in B_\kappa(q - \kappa, \omega, A)\}. \tag{13}$$

This is a standard problem, which has been extensively studied in the literature (see for instance Magill and Quinzii [28]). Assuming differentiability for $u(x)$, the first order conditions are

$$\tau \in \langle T_\kappa \rangle \text{ and } \pi_\kappa \in \langle T_\kappa \rangle^\perp \tag{14}$$

where $\pi_\kappa := \nabla u(x_\kappa^*)$. Note that by Proposition 1 (13) has a solution if and only if there is no arbitrage. Unfortunately there might be $\kappa \in b(-Z)$ where the decision problem will have no solution at all. In the sequel we thus have to restrict the set of admissible $\kappa \in b(-Z)$ to those which are consistent with a solution of the investor's problem, i.e. there exists a $\pi_\kappa \in \mathcal{R}_{++}^{S+1}$.

Normalizing this vector by the marginal utility of income at date 0 these conditions imply for security prices the relation

$$p = q - \kappa = \pi_{\kappa,\mathbf{1}} A, \ \forall z \in \mathcal{R}^J, \pi_{\kappa,1} \in \mathcal{R}^S_{++}. \tag{15}$$

From this equation it follows that the family of solutions will satisfy

$$q - \pi_{\kappa,\mathbf{1}} A \in b(-Z). \tag{16}$$

From Theorem 2, we know that any solution to the artificial, unconstrained problem found in this way, will automatically be feasible for the corresponding constrained problem with constraint set Z.

Now the contingent claim pricing problem can be attacked in the following way:

In $\mathcal{E} = \{u, \omega, A\}$ the formula

$$q - \kappa = \pi_{\kappa,\mathbf{1}} A$$

implies that for given security prices $q - \kappa$ the price of any income stream $y \in \langle A \rangle$ is given by

$$c^\kappa_q(y) = (q - \kappa)z \qquad \text{for any} \quad z \in \mathcal{R}^J \text{ such that} \quad y = Az$$

which implies a *family* of pricing formulas

$$c^\kappa_q(y) := \{(q - \kappa)z \mid \kappa \in b(-Z)\} = \pi_{\kappa,\mathbf{1}} y. \tag{17}$$

Finding the no arbitrage value of y in $\mathcal{E} = \{u, \omega, A, Z\}$ is then a solution to the problem

$$\begin{array}{ll} & \sup_\kappa (\pi_{\kappa,\mathbf{1}} y - \sigma_{-Z}(\kappa)) \\ & \quad \pi_{\kappa,\mathbf{1}} A = q - \kappa \\ \text{s.t.} & \quad \kappa \in b(-Z) \\ & \quad \pi_{\kappa,\mathbf{1}} \in \mathcal{R}^S_{++} \end{array} \tag{18}$$

The solution of (18) gives us an upper bound for the present value of y, but it does not guarantee that there actually is an investor who values y that much. This problem was clearly recognized by Chen [8] and Charupat and Prisman [7].

These remarks show that the approach suggested by Cvitanić and Karatzas [11] leads via a different route to the same results as the approach suggested here. In the light of our model, it becomes particularly transparent, why the barrier cone approach works. Moreover the analysis shows that it can be used as well *without* reference to a specific utility function and only invoking monotonicity. It is also *not* confined to representative investor setups. Once we know (q, A, Z) we can construct the artificial financial market $(q - \kappa, A)$. By standard theory, no arbitrage in the artificial market is equivalent to the existence of a strictly positive vector of state prices $\pi_\kappa \in \mathcal{R}^{S+1}_{++}$ such that $\pi_\kappa T_\kappa = 0$. Then for any income stream $y \in \mathcal{C}$ we can determine its arbitrage free value by solving (18). For the practically important cases where Z is polyhedral, this problem is a linear programming problem.

8 Conclusions

In this paper we have analyzed the pricing of contingent claims in security markets, where investors are constrained by various trading restrictions, which can be described as a convex set. We have given a characterization of no unlimited arbitrage in the *simplest* possible framework and have derived the implications for the valuation of an arbitrary contingent claim. The analysis shows that for problems, which are of particular relevance for practical purposes, this task can be achieved by solving a standard linear programming problem, which can be set up from the basic observable data of the economy. We have related this analysis to the optimal decision problem of an investor and have shown the various relations between the properties of an optimal solution to this problem and the arbitrage free values of contingent claims. This opens a unified view on the different approaches to asset pricing under portfolio constraints discussed in the literature and conveys their common underlying logic. We hope that these results will prove useful for financial economists who are interested to get an overview of the economics of portfolio constraints and security pricing, without going into the technicalities of continuous time stochastic finance. Ultimately, however, this is a paper about asset pricing. It is our hope that our results will prove useful for experts interested in practically developing and implementing valuation techniques inspired by tools from a heavenly linear space, within the constraints of an earthly convex set.

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9 Appendix

9.1 Figures

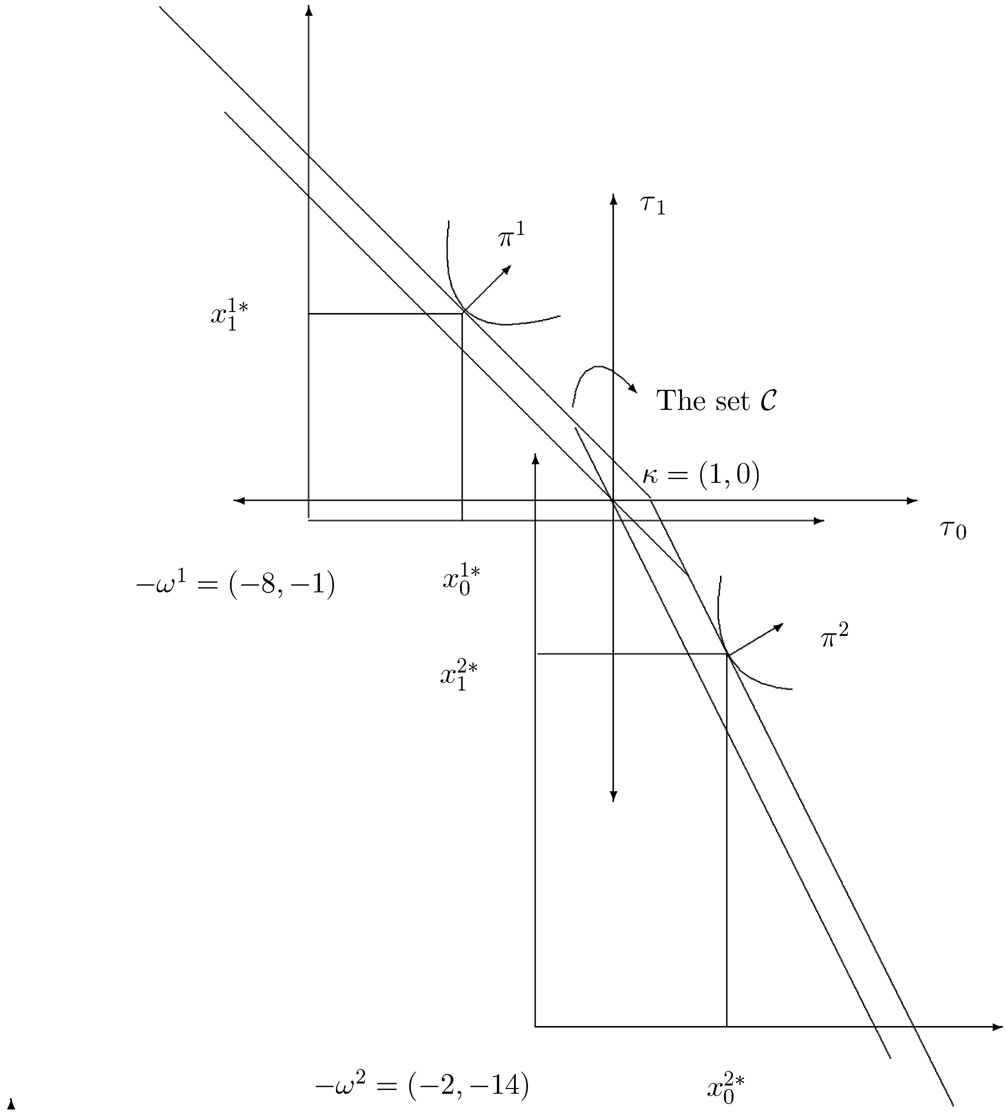

Figure 1: With portfolio constraints not all arbitrage opportunities are necessarily eliminated in equilibrium.

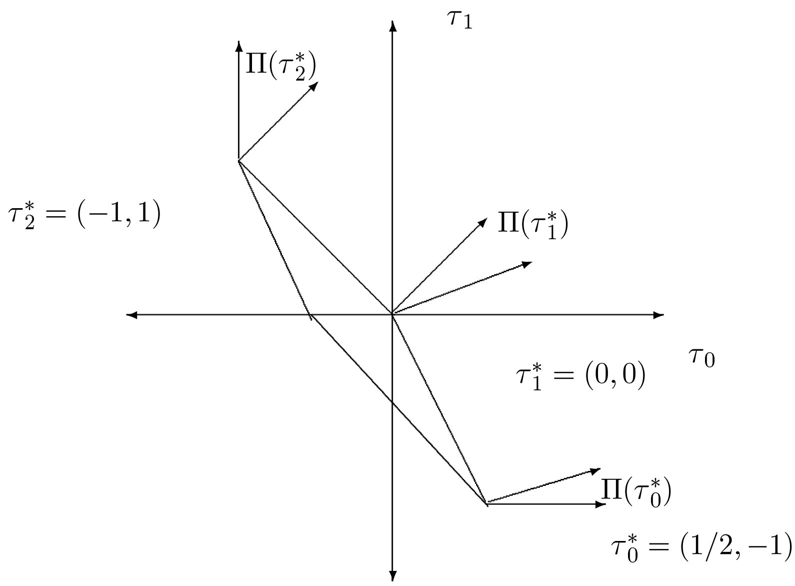

Figure 2: Rectangular Constraints

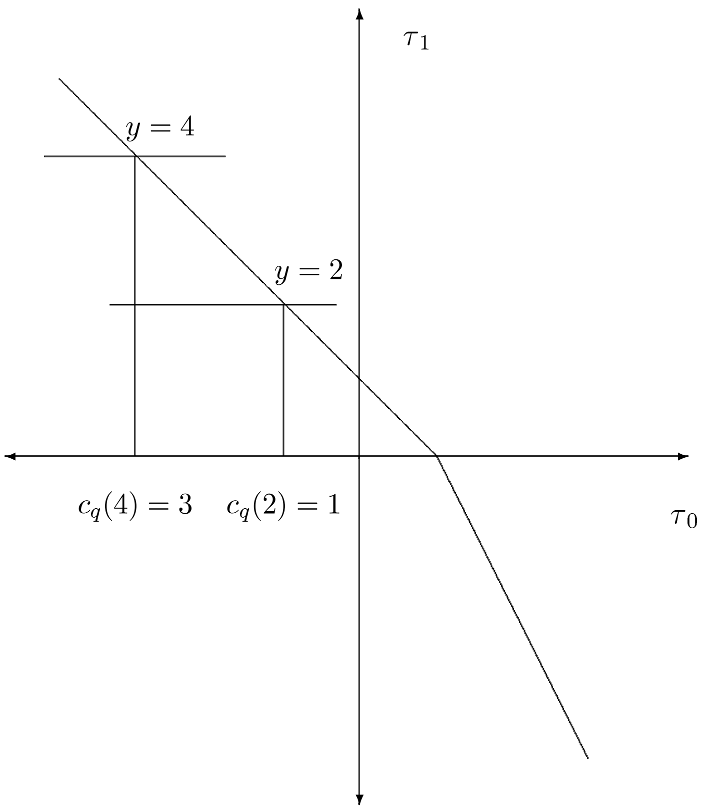

Figure 3: For general convex constraints the contingent claim pricing functional need not be sublinear.

9.2 Proofs

Lemma 1: *The set C of feasible income transfers is a convex set containing* 0.

Proof: $\mathcal{C} = \{\tau \in \mathcal{R}^{S+1} \mid \tau = Tz \, , \, z \in Z\} \subset \mathcal{R}^{S+1}$ by Definition 1. Since $0 \in Z$ and T is a linear transformation $\mathcal{C}$ must also contain 0, hence be non-empty. Let τ_1 and τ_2 be in $\mathcal{C}$ and consider $\lambda \in [0,1]$. Then $\lambda\tau_1 + (1-\lambda)\tau_2 = \lambda Tz_1 + (1-\lambda)Tz_2 = T(\lambda z_1 + (1-\lambda)z_2)$. Since Z is convex, $\lambda z_1 + (1-\lambda)z_2$ is in Z, thus $\lambda\tau_1 + (1-\lambda)\tau_2$ is in $\mathcal{C}$ hence $\mathcal{C}$ is convex. $\square$

Lemma 2: *If Z is a cone with vertex p, C is a cone with vertex Tp.*

Proof: If Z is a cone, $z \in Z$ implies that for all $\lambda \geqslant 0$ the vector $\lambda z \in Z$. Thus $Tz \in \mathcal{C}$ implies $T(\lambda z) = \lambda Tz \in \mathcal{C}$, thus $\mathcal{C}$ is a cone. If Z is a cone with vertex p, then $Z - p$ is a cone. Hence the set $\mathcal{C}' = \{\tau \in \mathcal{R}^{S+1} \mid \tau = Tx, x \in Z - p\}$ is a cone. As $\mathcal{C} = \mathcal{C}' + Tp$ it is a cone with vertex Tp. $\square$

Theorem 1: *(q, A, Z) admits no unlimited arbitrage if and only if there exists a vector $\bar{\pi} \in \mathcal{R}^{S+1}_{++}$ and a vector $\tau^* \in \mathcal{C}$ such that $\bar{\pi}\tau \leq \bar{\pi}\tau^*$, $\forall \tau \in \mathcal{C}$*

Proof: Let $\mathcal{T} = \{\tau^* \in \mathcal{C} \mid \nexists \, \tau \in \mathcal{C}$ such that $\tau > \tau^*\}$. Assume (q, A, Z) admits no unlimited arbitrage. Then $\mathcal{T}$ is not empty. Denote by $\mathcal{T}int(\mathcal{T})$ the interior of $\mathcal{T}$ with respect to the topology induced on $\mathcal{T}$, take some $\tau^* \in \mathcal{T}int(\mathcal{T})$ and consider the set $\mathcal{C}' = \{\tau \in \mathcal{R}^{S+1} \mid \tau + \tau^* \in \mathcal{C}\}$. Since we have assumed (NUA), $\mathcal{C}' \cap \mathcal{R}^{S+1}_{+} \backslash \{0\} = \emptyset$. $\mathcal{C}'$ is a non-empty, closed, convex, set in $\mathcal{R}^{S+1}$ since it is a translation of the set $\mathcal{C}$, which is non-empty, closed and convex by Lemma (1) and has a non-empty interior by (CONC). Let Δ be the non-negative simplex in $\mathcal{R}^{S+1}$. The simplex is a convex and compact subset of $\mathcal{R}^{S+1}$ containing no interior points of $\mathcal{C}'$ since we have assumed that there is no unlimited arbitrage. We can therefore apply a version of the separating hyperplane theorem (Magill and Quinzii [28, p. 73], p.73). The separation theorem implies that there is a linear functional $0 \neq \bar{\pi} \in \mathcal{R}^{S+1}$ such that

$$\sup_{\tau \in \mathcal{C}'} \bar{\pi}\tau < \inf_{\tau \in \Delta} \bar{\pi}\tau$$

Let $K(\mathcal{C}') = \{\lambda\tau \mid \tau \in \mathcal{C}'$ and $\lambda \in \mathcal{R}^{S+1}_{+}\}$ the convex cone generated by $\mathcal{C}'$. This cone is non-empty, closed and convex. As $0 \in \mathcal{T}int(\mathcal{T}) - \tau^*$ by assumption, $K(\mathcal{C}') \cap \mathcal{R}^{S+1}_{+} \backslash \{0\} = \emptyset$. Therefore the above separating hyperplane theorem is applicable. Because $\mathcal{C}' \subset K(\mathcal{C}')$, the set of $\pi's$ that separate $K(\mathcal{C}')$ from Δ also separate $\mathcal{C}'$ from Δ. Hence it suffices to show that there is a $\bar{\pi} \in \mathcal{R}^{S+1}_{++}$ separating $K(\mathcal{C}')$ from Δ such that $\bar{\pi}\tau \leq 0$ for all $\tau \in K(\mathcal{C}')$.

Since $0 \in K(\mathcal{C}')$, the separation inequality implies that $0 < \inf_{\tau \in \Delta} \bar{\pi}\tau$. Suppose now that $\bar{\pi}_s \leq 0$ for some state $s \in \mathcal{S}$ and consider the transfer $e_s = (0, ..., 0, 1, 0, ..., 0) \in \Delta$ which is 1 in state s and 0 otherwise. Then $0 < \inf_{\tau \in \Delta} \bar{\pi}\tau \leq \bar{\pi}e_s = \bar{\pi}_s \leq 0$ which is a contradiction. Hence $\bar{\pi} \in \mathcal{R}^{S+1}_{++}$. It remains to show that $\bar{\pi}\tau \leq 0$, $\forall \tau \in K(\mathcal{C}')$. Suppose there is a $\bar{\tau} \in K(\mathcal{C}')$ such that $\bar{\pi}\bar{\tau} > 0$. Since $K(\mathcal{C}')$ is a cone it follows that $\lambda\bar{\tau} \in K(\mathcal{C}')$, $\forall \lambda \geq 0$. It is easy to see that $\inf_{\tau \in \Delta} \bar{\pi}\tau = \min(\bar{\pi}_1, ..., \bar{\pi}_{S+1})$, which is finite. Now we can choose a sufficiently large λ such that $\lambda\bar{\pi}\bar{\tau} > \inf_{\tau \in \Delta} \bar{\pi}\tau$. This is a contradiction.

Contrary to the result for unrestricted economies we can not turn the inequality into an equality. The obvious reason is that $\tau \in \mathcal{C}'$ does *not* imply $-\tau \in \mathcal{C}'$. Thus we get only the inequality $\bar{\pi}\tau \leq 0$, $\forall \tau \in \mathcal{C}'$ which can be rewritten as $\bar{\pi}\tau \leq \bar{\pi}\tau^*$, $\forall \tau \in \mathcal{C}$.

To prove the other direction assume there is a $\tau^* \in \mathcal{C}$ and $\bar{\pi} \in \mathcal{R}^{S+1}_{++}$ such that $\bar{\pi}\tau \leq \bar{\pi}\tau^*$, $\forall \tau \in \mathcal{C}$. This implies $\bar{\pi}(\tau - \tau^*) \leq 0$ $\forall \tau \in \mathcal{C}$. Because $\bar{\pi} \in \mathcal{R}^{S+1}_{++}$ we can conclude that $\tau - \tau^* > 0$ is impossible for all $\tau \in \mathcal{C}$. And this is exactly the no unlimited arbitrage condition. □

Remark: We have imposed assumption (CONC) to cover a large class of practically important constraint situations. Unfortunately also some rather strange cases are also compatible with (CONC) and therefore we have to use the $\mathcal{T}int(\mathcal{T})$ construction. Why do we have to take $\tau^* \in \mathcal{T}int(\mathcal{T})$? If we took $\tau^* \in \mathcal{T}$ but not in $\mathcal{T}int(\mathcal{T})$, we could face the problem that $K(\mathcal{C}')$ meets $\mathcal{R}^{S+1}_{+}$ not only in 0 but also in other points $x \geq 0$ but not strictly larger than 0. Hence we would get the weaker result that there has to exist a $\bar{\pi} \in \mathcal{R}^{S+1}_{+}$ such that $\bar{\pi}\tau \leq \bar{\pi}\tau^*$, $\forall \tau \in \mathcal{C}$, where some components of $\bar{\pi}$ might be 0.

Corollary 1: *If Z is a cone with vertex p, the financial market (q, A, Z) admits no unlimited arbitrage if and only if there exists a vector $\bar{\pi} \in R^{S+1}_{++}$ such that $\bar{\pi}\tau \leq \bar{\pi}Tp$, $\forall \tau \in C$.*

Proof: It is sufficient to show that $\Pi(\tau) \subset \Pi(Tp)$ $\forall \tau \in \mathcal{C}$. If $\Pi(\tau^*) = \emptyset$ our assertion is trivially true. Now take any τ^* such that $\Pi(\tau^*) \neq \emptyset$ and suppose there exists a $\bar{\pi} \in \Pi(\tau^*)$ such that $\bar{\pi} \notin \Pi(Tp)$. This means that $\bar{\pi}Tp < \bar{\pi}\tau^*$. Now let $\tau^+ = (1+\varepsilon)(\tau^* - Tp) + Tp = \tau^* + \varepsilon(\tau^* - Tp)$ where $\varepsilon > 0$. By construction $\tau^+ \in \mathcal{C}$. Premultiplying τ^+ by $\bar{\pi}$ yields

$$\bar{\pi}\tau^+ = \bar{\pi}\tau^* + \varepsilon(\bar{\pi}\tau^* - \bar{\pi}Tp) > \bar{\pi}\tau^*$$

where the inequality follows from the assumption that $\bar{\pi} \notin \Pi(Tp)$. This result contradicts that $\bar{\pi} \in \Pi(\tau^*)$. Hence $\Pi(\tau^*) \subset \Pi(Tp)$. □

Remark: As Tp is not necessarily in $\mathcal{T}int(\mathcal{T})$ this Corollary implies that $\Pi(Tp) \neq \emptyset$ if there is a τ^* such that $\Pi(\tau^*) \neq \emptyset$., i.e. if the market admits no unlimited arbitrage.

Theorem 2: *If the financial market (q, A, Z) admits no unlimited arbitrage and there exists a $\bar{z} \in Z$ such that $A\bar{z} \geq \bar{y}$, then there exists a solution to the optimal hedging problem (8), i.e. there exists a $z^* \in Z$ such that $c_{\bar{q}}(\bar{y}) = \bar{q}z^*$ and $Az^* \geq \bar{y}$.*

Proof: Let $f(z) = \bar{q}z$. As f is a closed, convex and proper the recession function $f0^+$ of f is given by $(f0^+)(x) = \bar{q}x$.[16] The set of all vectors x such that $(f0^+)(x) = \bar{q}x \leq 0$ is called the recession cone of f. Now let $K = \left\{z \in \mathcal{R}^J \mid z \in Z \text{ and } Az \geq \bar{y}\right\}$. No unlimited arbitrage implies that $rec(f) \cap rec(K) = \{0\}$. Theorem 27.3. in Rockafellar (1970, p. 267) tells us that the fact that f and K have no direction of recession in common is sufficient to guarantee that the infimum is attained.

For the proof of Theorem 2 we apply a version of Fenchel's duality theorem [33, Corollary 31.2.1. p. 332]. The star symbol denotes conjugate (dual) operations. We denote by ψ_K the indicator function of a set K, by σ_K the support function of the set K and by $b(K)$ the barrier cone of the set.

Theorem: (Fenchel): *Suppose $X \subset \mathcal{R}^n$ and $W \subset \mathcal{R}^m$, that $B \in L(X, W)$ is a continuous, linear operator from X to W and that $f : X \to \mathcal{R} \cup \{\infty\}$ is a closed, proper, convex function*

[16] For a more general definition see Rockafellar (1970, p. 66).

and $g : W \to \mathcal{R} \cup \{\infty\}$ *is a closed proper concave function. Define*

$$\begin{aligned} v &:= \inf_{x \in X} [f(x) - g(Bx)] \\ v_* &:= \sup_{w \in W^*} [g^*(w) - f^*(B^*w)] \end{aligned}$$

$v = v_*$ *if either of the following conditions is satisfied:*

(a) *There exists an* $x \in ri(Dom f)$ *such that* $Bx \in ri(Dom g)$;[17]

(b) *There exists a* $w \in ri(Dom g^*)$ *such that* $B^*w \in ri(Dom f^*)$.

Under (a) the supremum is attained at some w *while under (b) the infimum is attained at some* x.

Proof: Rockafellar [33, Theorem 31.1., p. 327 and Corollary 31.2.1. p. 332]. □

Theorem 3: *If the financial market* $(\overline{q}, A, Z)$ *admits no unlimited arbitrage and either*

(a) Z *is polyhedral and there exists a* $\overline{z} \in Z$ *such that* $A\,\overline{z} \geqslant \overline{y}$ or

(b) *there exists a* $\hat{z} \in Z$ *such that* $A\,\hat{z} > \overline{y}$,

the dual problem

$$\max_{\pi_1 \in \mathcal{R}^S_+} (\pi_1 \overline{y} - \sup_{z \in Z}(-\overline{q}z + \pi_1 Az))$$

has a solution and the optimal values of the primal and the dual problems coincide.

Proof: Our primal problem is given by

$$\begin{array}{rc} & \inf_{z \in \mathcal{R}^J} \overline{q}z \\ s.t. & z \in Z \\ & Az - \overline{y} \in \mathcal{R}^S_+ \end{array} \tag{19}$$

Define the set $E = \{(z, y) \mid z \in Z \text{ and } y - \overline{y} \in \mathcal{R}^S_+\}$ and denote by I the $J \times J$ identity matrix. Using this matrix we form the new $(J + S) \times J$ matrix

$$B = \begin{bmatrix} I \\ A \end{bmatrix}$$

Thus we have $z \in Z$ and $Az - \overline{y} \in \mathcal{R}^S_+ \Leftrightarrow Bz \in E$. Now let ψ_E denote the indicator function of E. Thus we can rewrite the primal problem by

$$\inf_{z \in \mathcal{R}^J} [\overline{q}z + \psi_E(Bz)]$$

The function f in the theorem corresponds to $\overline{q}z$, the function g in the theorem corresponds to $-\psi_E(Bz) := g(Bz)$ the operator A corresponds to our matrix B. $f(z)$ is closed proper and

[17] ri denotes the relative interior of a set [33, p. 44].

convex and by (CONC) the indicator function ψ_E is also closed, proper and convex and hence $-\psi_E$ is closed, proper and concave. (This follows from Aubin, Corollary 1.1. p. 13)

To derive the dual, we have to find the conjugate functions. The conjugate function to $f(z) := \bar{q}z$ is $f^*(\kappa) = \psi_{\{\bar{q}\}}(\kappa)$. The conjugate function $g^*(\kappa, \pi) = \inf_{(z,y)\in\mathcal{R}^{J+S}}(\kappa z + \pi y + \psi_E(z, y))$. This function can be rewritten as

$$\inf_{z\in\mathcal{R}^J}(\kappa z + \psi_Z(z)) + \inf_{y\in\mathcal{R}^S}(\pi y + \psi_{\mathcal{R}^S_+ + \bar{y}}(y))$$

or equivalently

$$-\sup_{z\in\mathcal{R}^J}(-\kappa z - \psi_Z(z)) + \inf_{y\in\mathcal{R}^S}(\pi y + \psi_{\mathcal{R}^S_+ + \bar{y}}(y))$$

or equivalently

$$-\sigma_{-Z}(\kappa) + \inf_{y\in\mathcal{R}^S}(\pi y + \psi_{\mathcal{R}^S_+ + \bar{y}}(y)).$$

The set $\mathcal{R}^S_+ + \bar{y} = \{y \in \mathcal{R}^S \mid y - \bar{y} \in \mathcal{R}^S_+\} = \{x + \bar{y} \in \mathcal{R}^S \mid x \in \mathcal{R}^S_+\}$. Thus we get

$$g^*(\kappa, \pi) = -\sigma_{-Z}(\kappa) + \pi\bar{y} + \inf_{x\in\mathcal{R}^S}(\pi x + \psi_{\mathcal{R}^S_+}(x))$$

Obviously we have

$$\inf_{x\in\mathcal{R}^S}(\pi x + \psi_{\mathcal{R}^S_+}(x)) = \begin{cases} 0 & \text{if } \pi \geq 0 \\ -\infty & \text{else} \end{cases}$$

We thus conclude that

$$g^*(\kappa, \pi) = -\sigma_{-Z}(\kappa) + \pi\bar{y}$$

where $\text{Dom}g^* = b(-Z) \times \mathcal{R}^S_+$. The conjugate of the operator B is simply the transposed matrix B^T. Now with these objects by applying the theorem, we can formulate the dual as

$$\sup_{\kappa,\pi}[-\sigma_{-Z}(\kappa) + \pi\bar{y} - \psi_{\{\bar{q}\}}(\kappa + \pi A)]$$

Now clearly

$$\psi_{\{\bar{q}\}}(\kappa + \pi A) = \begin{cases} 0 & \text{if } \bar{q} = \kappa + \pi A \\ \infty & \text{else} \end{cases}$$

Thus if there is a finite solution it must hold that

$$\bar{q}z = \kappa z + \pi A z \quad \forall z \in \mathcal{R}^J$$

Thus we can write the dual problem as:

$$\begin{array}{rc} & \sup\,[\pi\bar{y} - \sigma_{-Z}(\kappa)] \\ s.t. & \pi \in \mathcal{R}^S_+ \\ & \kappa = \bar{q} - \pi A \in b(-Z) \end{array}$$

To prove the assertion we have to show that assumption (a) or (b) of the theorem are sufficient. In (a) it is assumed that there exists a $\bar{z} \in Z$ such that $A\,\bar{z} \geqslant \bar{y}$. This means that there exists a

$z \in \text{Dom}f$ such that $Bz \in \text{Dom}g$. As g is polyhedral by assumption (a), we may apply Fenchel's Theorem. In (b) it is assumed that there exists a $\hat{z} \in Z$ such that $A\ \hat{z} > \bar{y}$. Now the convexity of Z implies that there is a $z^* \in ri(\text{Dom}f) = ri(\mathcal{R}^J)$ such that $z^* \in ri(\text{Dom}g) = ri(E)$. And again we can apply Fenchel's Theorem. □

Corollary 3: *If the financial market (q, A, Z) admits no unlimited arbitrage and Z is a convex cone with vertex p and there exists a $\bar{z} \in Z$ such that $A\bar{z} \geq \bar{y}$ the optimal hedging problem (8) has a solution. The dual problem can be stated as:*

$$\begin{array}{rl} & \max\ (\pi_{\mathbf{1}} y + (q - \pi_{\mathbf{1}} A)p) \\ s.t. & (1, \pi_{\mathbf{1}}) \in (\mathcal{C} - Tp)^{\ominus} \cap \mathcal{R}_+^{S+1} \end{array}$$

If either condition (a) or (b) of Theorem 3 hold, the dual problem is solvable.

Proof: When Z is a cone with vertex p, $\sigma_{-Z}(\kappa) = -\kappa p$. We have to show that the requirement that $q - \pi_{\mathbf{1}} A \in b(-Z)$ is equivalent to the requirement that $(1, \pi_{\mathbf{1}}) \in (\mathcal{C} - Tp)^{\ominus}$. Assume $q - \pi_{\mathbf{1}} A \in b(-Z)$. Then $\sigma_{-Z}(q - \pi_{\mathbf{1}} A) = -(q - \pi_{\mathbf{1}} A)p$. Thus

$$(-q + \pi_{\mathbf{1}} A)z \leq (-q + \pi_{\mathbf{1}} A)p \quad \forall z \in Z$$

This is the same as

$$(1, \pi_{\mathbf{1}}) \begin{pmatrix} -q \\ A \end{pmatrix} z \leq (1, \pi_{\mathbf{1}}) \begin{pmatrix} -q \\ A \end{pmatrix} p \quad \forall z \in Z$$

which is the same as

$$(1, \pi_{\mathbf{1}})\tau \leq (1, \pi_{\mathbf{1}})Tp \quad \forall \tau \in \mathcal{C}$$

and this is what we wanted to show. □

Corollary 5: *If Z is a cone then the pricing functional $c_{\bar{q}}(y)$ is sublinear.*

Proof: If Z is a cone then $c_q(y) = \max_{\pi_{\mathbf{1}} \in \mathcal{R}_+^S} \pi_{\mathbf{1}} y$. Obviously $\max_{\pi_{\mathbf{1}} \in \mathcal{R}_+^S} \pi_{\mathbf{1}} \alpha y = \alpha \max \pi_{\mathbf{1}} y$ for any $\alpha \geqslant 0$. Thus the pricing functional is homogeneous. Subadditivity follows from

$$\max_{\pi_{\mathbf{1}} \in \mathcal{R}_+^S} (\pi_{\mathbf{1}}(y + y')) = \max_{\pi_{\mathbf{1}} \in \mathcal{R}_+^S} (\pi_{\mathbf{1}} y + \pi_{\mathbf{1}} y') \leq \max_{\pi_{\mathbf{1}} \in \mathcal{R}_+^S} \pi_{\mathbf{1}} y + \max_{\pi_{\mathbf{1}} \in \mathcal{R}_+^S} \pi_{\mathbf{1}} y'$$

which is a triangle inequality. □

Proposition 1: *Problem (11) has a solution if an only if there is no unlimited arbitrage in the financial market.*

Proof: Assume first that (11) has a solution x^{i*}, so that

$$x^{i*} \in \arg\max \left\{ u^i(x^i) \mid x^i \in B^i(q, \omega^i, A, Z) \right\}$$

with $x^{i*} - \omega^i = Tz^{i*}$. If there is unlimited arbitrage we can find a $z^i \in Z$ such that $Tz^i > x^{i*} - \omega^i$. Then $x^i = \omega^i + Tz^i > x^{i*}$. Since u^i is strictly monotone, this implies $u^i(x^i) > u^i(x^{i*})$. Thus x^{i*} can not be optimal.

Now assume that there is no unlimited arbitrage. By Theorem 1 there exists then a strictly positive vector of state prices with $\bar{\pi} \in \mathcal{R}_{++}^{S+1}$ and a vector $\tau^* \in \mathcal{C}$ such that $\bar{\pi}\tau \leq \bar{\pi}\tau^*$, $\forall \tau^i \in$

$\mathcal{C}$. Take this vector $\bar{\pi}$ and form a new set $B^i(\bar{\pi}, \omega^i) = \left\{x^i \in \mathcal{R}_+^{S+1} \mid \bar{\pi}(x^i - \omega^i) \leq \bar{\pi}\tau^*\right\}$. Let $m^i = \bar{\pi}(\tau^* + \omega^i)$. Then $x^i \in B^i(\bar{\pi}, \omega^i)$ implies $0 \leq x_s^i \leq m^i/\bar{\pi}_s$, $s = 0, 1, ...S$. Therefore $B^i(\bar{\pi}, \omega^i)$ is clearly bounded and since the inner product is continuous, it is also closed. Because it is a subset of $\mathcal{R}^{S+1}$ it is compact by the Heine-Borel Theorem. For any $x^i \in B^i(q, \omega^i, A, Z)$ there is a $z^i \in Z$ with $x^i - \omega^i \leq Tz^i$. Because there is no unlimited arbitrage $\bar{\pi}\tau \leq \bar{\pi}\tau^*$ $\forall \tau^i \in \mathcal{C}$, we have that $\bar{\pi}(x^i - \omega^i) \leq \bar{\pi}\tau^*$, $\forall \tau^i \in \mathcal{C}$ so $x^i \in B^i(\bar{\pi}, \omega^i)$. This argument implies that $B^i(q, \omega^i, A, Z) \subset B^i(\bar{\pi}, \omega^i)$. By (CONC) and Lemma 1, $B^i(q, \omega^i, A, Z)$ is closed. Since $B^i(\bar{\pi}, \omega^i)$ is compact, this implies that $B^i(q, \omega^i, A, Z)$ is compact, because a closed subset of a compact set is compact. Since $u^i(x^i)$ is continuous, the existence of a solution to (11) follows from the Weierstrass theorem. □

Proposition 2: *Assume u^i is also differentiable. Thus $\nabla u^i(x^i) > 0$ $\forall x^i \in B^i(q, \omega^i, A, Z)$. If x^{i*} is a solution to (11) then $\nabla u^i(x^{i*}) \in N_{B^i}(x^{i*})$ and $\lambda \nabla u^i(x^{i*})$ is a state price vector for $\lambda > 0$.*

Proof: u^i is continuous, strictly monotone and quasiconcave. By (CONC) $B^i(q, \omega^i, A, Z)$ is non-empty and convex. Thus (11) is a convex programming problem. By a theorem from convex analysis (see for instance Magill and Quinzii 1996, p. 410) it follows therefore that (11) has a solution if and only if

$$\begin{aligned} x^{i*} &\in B^i(q, \omega^i, A, Z) \\ \nabla u^i(x^{i*}) &\in N_{B^i}(x^{i*}). \end{aligned}$$

Since u^i is strictly monotone, it follows that $\nabla u^i(x^{i*}) \in \mathcal{R}_{++}^{S+1}$. Now we have to show that $\nabla u^i(x^{i*})$ is indeed a state price vector. In order to do so, first observe that $x^{i*} - \omega^i$ is such that there is no $\tau \in \mathcal{C}$ with $\tau > x^{i*} - \omega^i = \tau^*$ because otherwise x^{i*} could not be a solution to the maximization problem. The second important observation is that the normal cone to B^i at x^{i*} is identical to the closure of the set of all $\bar{\pi} \in \mathcal{R}_{++}^{S+1}$ which solve the no unlimited arbitrage inequality $\bar{\pi}\tau \leq \bar{\pi}\tau^*$ for all $\tau \in \mathcal{C}$, i.e., $N_{B^i}(x^{i*}) = cl(\Pi(x^{i*} - \omega^i)) = cl(\Pi(\tau^*))$. To see this, consider the subset of the budget set $D^i(q, \omega^i, A, Z) := \left\{x^i \in \mathcal{R}_+^{S+1} \mid x^i - \omega^i = Tz^i,\ z^i \in Z\right\}$. Because $u^i(x)$ is strictly monotone, $x^{i*} \in D^i(q, \omega^i, A, Z)$. Now clearly $D^i \subset B^i$ and therefore $N_{B^i}(x^{i*}) \subset N_{D^i}(x^{i*})$. Since $D^i(q, \omega^i, A, Z) = \omega^i + \mathcal{C}$ the normal cone to $D^i(q, \omega^i, A, Z)$ at x^{i*} is the same as the normal cone to $\mathcal{C}$ at $\tau^* = x^{i*} - \omega^i$, i.e. $N_{D^i}(x^{i*}) = N_{\mathcal{C}}(x^{i*} - \omega^i) = N_{\mathcal{C}}(\tau^*)$. According to our previous definition $\Pi(\tau^*) = N_{\mathcal{C}}(\tau^*) \cap \mathcal{R}_{++}^{S+1}$. Since $\mathcal{C}$ is closed $N_{\mathcal{C}}(\tau^*)$ is closed. Therefore $cl(\Pi(\tau^*)) = N_{\mathcal{C}}(\tau^{i*}) \cap \mathcal{R}_+^{S+1}$. To prove that $N_{B^i}(x^{i*}) \subset cl(\Pi(\tau^*)) = N_{D^i}(x^{i*}) \cap \mathcal{R}_+^{S+1}$ it suffices to show that any $\pi \in N_{B^i}(x^{i*})$ is nonnegative. Suppose to the contrary that there is a $\pi \in N_{B^i}(x^{i*})$ such that $\pi_s < 0$ for some index s. Because $\pi \in N_{B^i}(x^{i*})$ we know by the definition of a normal cone that

$$\pi(x^{i*} - x^i) \geq 0 \ \forall x^i \in B^i.$$

Now we choose $x = x^{i*} - e_s$. Of course $x \in B^i$. But $\pi(x^{i*} - x) = \pi_s < 0$. This is a contradiction to the assumption $\pi \in N_{B^i}(x^{i*})$. Therefore any $\pi \in N_{B^i}(x^{i*})$ is nonnegative and $N_{B^i}(x^{i*}) \subset cl(\Pi(\tau^*))$. To prove that $cl(\Pi(\tau^*)) = N_{D^i}(x^{i*}) \cap \mathcal{R}_+^{S+1} \subset N_{B^i}(x^{i*})$ suppose there is a nonnegative $\pi \in N_{D^i}(x^{i*})$ which is not in $N_{B^i}(x^{i*})$. This means there is an $x_{B^i} \in B^i$ such that $\pi(x^{i*} - x_{B^i}) < 0$. For any $x_{B^i} \in B^i$ there exists a $x_{D^i} \in D^i$ such that $x_{B^i} \leq x_{D^i}$. As the chosen π is nonnegative this implies $\pi x_{B^i} \leq \pi x_{D^i}$. But this would mean that $\pi x^{i*} < \pi x_{D^i}$ and this is a contradiction to the assumption that $\pi \in N_{D^i}(x^{i*})$. So we may conclude $N_{B^i}(x^{i*}) = cl(\Pi(\tau^*))$. Since $\nabla u^i(x^{i*}) \in \mathcal{R}_{++}^{S+1}$ we see that $\nabla u^i(x^{i*}) \in \Pi(\tau^*)$. Thus $\lambda \nabla u^i(x^{i*})$ is a state price vector for any $\lambda > 0$ and this is what we wanted to show. □

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