OESTERREICHISCHE NATIONALBANK
EUROSYSTEM

WORKING PAPER 251

Housing and the secular decline in real interest rates

Markus Knell

The *Working Paper series of the Oesterreichische Nationalbank* is designed to disseminate and to provide a platform for discussion of either work of the staff of the OeNB economists or outside contributors on topics which are of special interest to the OeNB. To ensure the high quality of their content, the contributions are subjected to an international refereeing process. The opinions are strictly those of the authors and do in no way commit the OeNB.

The Working Papers are also available on our website (http://www.oenb.at) and they are indexed in RePEc (http://repec.org/).

Publisher and editor *Oesterreichische Nationalbank*
Otto-Wagner-Platz 3, 1090 Vienna, Austria
PO Box 61, 1011 Vienna, Austria
www.oenb.at
oenb.info@oenb.at
Phone (+43-1) 40420-6666
Fax (+43-1) 40420-046698

Editor *Martin Summer*

Cover Design *Information Management and Services Division*

DVR 0031577

ISSN 2310-5321 (Print)
ISSN 2310-533X (Online)

Housing and the secular decline in real interest rates*

Markus Knell**
Oesterreichische Nationalbank

December 2023

Abstract

In this paper, I study the role of housing for wealth accumulation and the determination of the equilibrium real interest rate within a continuous-time overlapping generations model that incorporates a realistic demographic structure and households that save for life-cycle and bequest reasons. The benchmark model contains three groups of dwellers: renters, homeowners with mortgages and outright owners. The latter group is assumed to inherit their dwellings, to use them as lifelong residences and to bequest them to their descendants. In addition there is also the group of the top 1% who are assumed to have higher incomes and stronger bequest motives. The calibrated model predicts a decline in the equilibrium real interest rate between 1980 and 2018 of almost 4 percentage points (pp), an increase in the wealth-to-income ratio of almost 250 pp and an increase in the share of housing wealth of almost 8 pp. All of these patterns are broadly in line with the empirical observations. In addition, the results of the model also align with other empirical regularities, like the mute response in the capital-to-income ratio, the trend in inheritance flows and the proliferation of mortgages. The paper closes with a discussion of why the assumptions about the behavior of outright owners are crucial for capturing these developments.

Keywords: Housing, Wealth, Interest Rate, Saving
JEL-Classification: D14, D31, G51, R21

*I thank an anonymous referee and participants at various conferences for valuable comments and suggestions. The views expressed in this paper do not necessarily reflect those of the Oesterreichische Nationalbank.

**OeNB, Economic Studies Division, Otto-Wagner-Platz 3, POB-61, A-1011 Vienna; Phone: (++43-1) 40420 7218, email: markus.knell@oenb.at.

Non-technical summary

In recent decades, real interest rates have experienced a consistent decline by about 3 percentage points (pp). Various factors such as shifts in savings patterns, technological advancements, and income inequality have been explored as potential drivers for this trend. So far, however, the existing literature on this topic has neglected the role of land and real estate as a key asset class. This is particularly noteworthy given the historical importance of real estate for global wealth.

This paper aims to address this gap by introducing a housing sector into a standard economic model. The central question it seeks to answer is how the inclusion of a housing sector influences the determination of equilibrium interest rates and their evolution over time. The presence of a housing sector may serve as a "reservoir" for excess savings, thus mitigating their impact on the capital stock and the resulting downward pressure on interest rates. Moreover, it may alter the relative importance of various influential factors discussed above, such as demographics, inequality, and technological advancements.

To capture the life-cycle dimension of savings and housing decisions, the study employs an overlapping generations model in continuous time, building on the work of Piketty (2011). This model considers households working in the first part of their lives and receiving pay-as-you-go pension benefits during retirement in their second part. The utility function of households incorporates intratemporal, intertemporal, and intergenerational elements, resulting in a savings schedule guided by both life-cycle and bequest motives.

The calibrated model considers four distinct societal groups, distinguishing between the top 1% and the rest of society. The bottom 99% is further segmented into renters, owner-occupiers with mortgages, and outright owners. These outright owners are assumed to inherit their dwellings, to use them as their residences throughout life, and to pass them on to their descendants.

The model is solved numerically, where I follow Summers & Rachel (2019) and Platzer & Peruffo (2022) and focus on steady-state comparisons between an "initial period" (roughly around 1980, before the decline in real interest rates) and a "current period" (approximately around 2018), considering changes in various economic parameters.

Results indicate that the inclusion of a housing sector tend to increase equilibrium interest rates, reflecting the additional investment opportunity for household savings. The steady state comparison, on the other hand, implies a decline in the interest rate between 1980 and 2018 by almost 4 pp (from 9.6% to 5.7%), an increase in the wealth-to-income ratio by almost 250 pp (from 350% to 599%) and an increase in the share of housing wealth in total wealth by 8 pp (from 46% to 54%). These results are broadly in line with the empirically observed data. What is more, the extended model is also able to account for a number of other empirical regularities like the wedge between safe and risky interest rates, for an increase in the inheritance flows and for a proliferation of mortgages. Models that exclude housing have difficulties to match all of these features at the same time as discussed at the end of the paper.

1 Introduction

Real interest rates have exhibited a constant decline over the recent decades. A seminal study by Laubach & Williams (2003) documented a decrease of approximately 3 percentage points (pp) in the long-run real interest rate in the United States since 1980. This decline was subsequently corroborated by various studies, including Summers & Rachel (2019), who estimated a similarly sized decrease in the *global* real interest rate since 1980. This empirical trend has given rise to a substantial body of literature aiming to elucidate its underlying causes. The primary explanatory factors highlighted in this literature include shifts in savings patterns and advancements in technological progress. Excess savings may stem from demographic aging, which increases the need to prepare for old age (Eggertsson et al. 2019, Auclert et al. 2021), or they may result from the widespread increase in income inequality, coupled with income-dependent savings rates (Mian et al. 2021*b*). Another strand of the literature emphasizes the significance of a global savings glut (Bernanke 2005) and a shortage of safe assets which had a dampening effect on government bond rates in advanced economies (Caballero et al. 2017). Additionally, some scholars have argued that technological developments, particularly the productivity slowdown, may have contributed to the decline in interest rates (Gordon 2014). A comprehensive discussion of various explanatory channels can be found in Rachel & Smith (2015) and Mian et al. (2021*a*). Summers & Rachel (2019) and Platzer & Peruffo (2022) employ large-scale quantitative models to assess the relative importance of these different channels in explaining the decline in interest rates.

What is common to the existing literature on this topic, however, is the neglect of land and real estate as an asset class, with the assumption that physical capital is the only asset available to absorb the volume of savings. This is a non-trivial omission given that real estate constitutes the most important component of global wealth—a fact that has been true throughout history. This paper aims to address this gap by introducing a housing sector into an otherwise standard economic model. The primary question it seeks to answer is how this extension influences the determination of equilibrium interest rates and their evolution over time. The presence of a housing sector may serve as a "reservoir" for excess savings, thus mitigating their impact on the capital stock and the resulting downward pressure on interest rates. Moreover, it may alter the relative importance of various influential factors discussed above, such as demographics, inequality, and technological advancements.

To capture the life-cycle dimension of savings and housing decisions, I employ an overlapping generations (OLG) model in continuous time, that is based on Piketty (2011). In this framework, households work for the first part of their lives while being retired in the second part, during which they also receive pay-as-you-go (PAYG) pension benefits. The utility function of households comprises three elements: an intratemporal (housing vs. non-housing consumption), an intertemporal (expenditures over time), and an intergenerational (individual utility vs. a warm-glow bequest motive) element. This structure results in a savings schedule that is guided by both a life-cycle motive (shaped by the generosity of the public pension system) and a bequest motive (affected by the strength of intergenerational ties). In the benchmark model I assume that society is composed of four groups that differ along various dimensions. First, I explicitly distinguish between the top 1% and the rest of society, where the former are characterized by higher lifetime incomes and more pronounced bequest motives. Second, I assume that the bottom 99% consist of three groups of dwellers: renters, owner-occupiers with mortgages and outright owners.[1] The latter groups of owner-occupiers are specified in a stylized fashion in order to keep the model tractable. In particular, members of the first group of owner-occupiers are assumed to finance their home purchases entirely with mortgages and to furthermore continuously adapt and refinance these purchases. In contrast, outright owners are assumed to inherit their dwellings, to use them as their residences throughout life, and to pass them on to their descendants. The houses of outright owners are thus never on the market, but they are nevertheless valued at the prevailing house prices, thereby contributing to housing wealth. Altogether, the financial savings accumulated by households collectively give rise to a wealth supply schedule, that depends positively on the interest rate.

On the other side, there are three asset categories where these aggregate savings can be allocated: government bonds, physical capital, and the housing stock. Capital demand arises from the assumption of a Cobb-Douglas production function and competitive markets for goods and factors of production, implying an inverse relationship between capital demand and the interest rate. Following the asset-market framework for the housing sector (cf. Poterba 1984), the purchase price of the housing stock is determined by the present value of the discounted stream of rental income (both actual and imputed). Consequently, the demand by the housing sector also depends negatively on the interest rate and positively on the equilibrium rent. In summary, the equilibrium interest rate is determined by the intersection of the wealth supply schedule (reflecting households' accumulated

[1] It is empirically well-documented that the type of tenure (renting or owning) and the source of income (labor or capital) are important determinants of the position in the wealth distribution. See, e.g., Fessler & Schürz (2021).

wealth) and the wealth demand schedule (encompassing investments in physical capital, the housing stock, and government bonds).

The model is solved numerically, where I follow Summers & Rachel (2019) and Platzer & Peruffo (2022) and focus on steady-state comparisons between an "initial period" (roughly around 1980, before the decline in real interest rates) and a "current period" (approximately around 2018). For the calibration I use data from the Worldbank and from the OECD that refer to the group of high income countries around these two dates. Furthermore, I calibrate the bequest motives to target a wealth-to-income ratio of 350% and a share of wealth held by the top 1% of 28% (both in the initial period).[2]

The benchmark model implies an initial interest rate of 9.6% which is associated with the targeted initial wealth-to-income ratio of 350%. These values are higher than those obtained in a model without housing (8.4% and 238%, respectively) which reflects the fact that the existence of housing provides another investment vehicle for the supply of household savings thereby lowering investments into the capital stock and driving up the equilibrium interest rate. In a next step I calibrate the model to the situation around 2018 using parameter values that reflect the changes in inequality, productivity growth, population growth, life expectancy, the retirement system, and in the housing market (in particular involving a decrease in the share of renters and an increase in the share of houses in possession of the outright owners). The implied steady-state is characterized by a decline in the interest rate by almost 4 pp (from 9.6% to 5.7%), an increase in the wealth-to-income ratio by almost 250 pp (from 350% to 599%) and an increase in the share of housing wealth in total wealth by 8 pp (from 46% to 54%). These results are broadly in line with the empirically observed data (with the implied decline in the real interest rate being on the larger side).[3]

In addition to these core variables the model also has implications for a number of further macroeconomic variables that can be compared to their real-world counterparts in order to assess the plausibility of the benchmark model and the differences to the competing models. In particular, the model implies an only modest increase in the capital-to-income ratio (from 168% to 204%), a 2.6 pp rise in inheritance flows (from 6.6% to

[2]These targets follow from the data reported in Piketty & Zucman (2014) and Bauluz et al. (2022) and in Alvaredo et al. (2018), respectively, which are discussed more extensively below.

[3]As described in a later section, Piketty & Zucman (2014) report for the period from 1970-2010 an average increase in the wealth-to-income ratio from 355% to 465% while Bauluz et al. (2022) find for the period from 1980-2018 an average increase from 301% to 537%. For the share of housing wealth, on the other hand, the empirical literature reports an average increase from around 36% to 53% (Piketty & Zucman 2014, Alvaredo et al. 2018).

9.2%), and a 38 pp increase in the share of mortgages-to-GDP (from 36% to 74%). These additional results align with empirical data reported in Bonnet et al. (2014), Alvaredo et al. (2017) and Jordà et al. (2016), respectively, with the latter, for example, indicating that mortgages have increased from about 20% of GDP at the start of the 20th century to approximately 70% today. Furthermore, the model gives rise to aggregate savings rates that are broadly in line with the empirical observed data, in particular if one excludes capital gains from the definition of savings (as should be done when seeking a comparison to official data). Finally and connected to the last issue, it is worth noting that the behavior of outright owners, who simply maintain their inherited houses, is consistent with recent findings from Norway. In particular, Fagereng et al. (2019) document that saving rates net of capital gains remain roughly constant across the wealth distribution, while saving rates including capital gains systematically increase with wealth.

A later part of the paper explains why the assumption about the behaviour of outright owners is crucial in order to capture the observed increase in the share of housing wealth and the behavior of other macroeconomic magnitudes. In particular, outright owners are assumed to simply keep their housing stock constant even if house prices are increasing due to other developments in the economy (e.g. the decline in interest rates). This tends to push up total wealth and the share of housing wealth. I show that both a model without outright owners and a model with only renters fall short along this important dimension since both would imply a *decrease* in the share of housing wealth. A model without housing, furthermore, implies high savings rates and large increases in the capital-to-income ratio which are at odds with what is empirically observed. In an appendix I also provide a simple analytical example that demonstrates why and under which conditions an increase in the ownership structure leads to an elevation in the total wealth-to-income ratio and the share of housing wealth. This analytical insight further corroborates the main results of the calibrated model.

Related literature: The main literature about the decline in the equilibrium real interest rate has been summarized at the beginning of the introduction. Notably, the quantitative models developed by Summers & Rachel (2019) and Platzer & Peruffo (2022) can be directly compared to the findings presented in this paper. A subsequent section in this paper demonstrates a substantial alignment between these studies. Specifically, all three investigations identify the rise in income inequality, the slowdown in productivity growth and the increase in life expectancy as the primary drivers behind the decrease in interest rates. The second area of related literature focuses on the components of wealth and their evolution over time. Notable contributions in this domain include Piketty (2011), Piketty & Zucman (2014) and Bauluz et al. (2022). Within this context, Bonnet et al. (2014), Bonnet et al. (2021) and Rognlie (2016) have discussed the role of housing in explaining observed trends in aggregate wealth. The long-term trajectory of house prices and the parallel increase in household mortgage lending are documented in Knoll et al. (2017) and Jordà et al. (2016). Furthermore, the paper is also related to the literature on housing and macroeconomics. The majority of research in this strand of research focuses on the issue of short-run fluctuations, with a particular emphasis on developments following the onset of the Great Financial Crisis (Favilukis et al. 2017, Justiniano et al. 2019, Kaplan et al. 2020). Papers that deal with long-run developments are Borri & Reichlin (2018), Grossmann, Larin, Löffiad & Steger (2021) and Grossmann, Larin & Steger (2021). The latter paper presents a complementary explanation for the increase in the share of housing wealth, centered on a model that distinguishes between land and structures and in which productivity increases are weaker in the construction sector than in the non-housing sector. An encompassing literature review is provided by Piazzesi & Schneider (2016).

The paper is structured as follows. The supply side of the model is presented in the subsequent section with the elaboration of the demand side following in section 3. The general solution is expounded in section 4, while the numerical results are detailed in section 5. Section 6 concludes and a number of appendices contain additional derivations and extensions.

2 Supply side

2.1 Non-housing production

Total output of non-housing (or "normal" or "numeraire") goods and services Y_{Nt} is assembled by a standard Cobb-Douglas production function:

$$Y_{Nt} = F(K_t, \mathcal{A}_t L_t) = K_t^{\alpha} (\mathcal{A}_t L_t)^{1-\alpha}. \tag{1}$$

Production uses physical capital K_t and the aggregate labor supply L_t where average labor productivity $\mathcal{A}_t$ is assumed to grow at a constant rate g:

$$\mathcal{A}_t = \mathcal{A}_0 e^{gt}$$

with $\mathcal{A}_0$ given. It is assumed that the total population and labor supply grow at rate n, i.e. $L_t = L_0 e^{nt}$ (the details of the demographic structure are described in section 3.1).

The proceeds of the "normal" production are divided between aggregate capital income Y_{Kt} and aggregate labor income Y_{Lt}, i.e. $Y_{Nt} = Y_{Lt} + Y_{Kt}$. Factor markets are assumed to be competitive and it thus holds that $Y_{Lt} = (1-\alpha)Y_{Nt}$ and $Y_{Kt} = \alpha Y_{Nt}$. The net return on capital is denoted by r_{kt}, i.e.:

$$r_{kt} = \frac{\partial Y_{Nt}}{\partial K_t} - \delta_k = \alpha \frac{Y_{Nt}}{K_t} - \delta_k, \tag{2}$$

where δ_k stands for the rate of capital depreciation.

2.2 Housing

The total housing stock is denoted by $\overline{H}_t$. It is plausible to assume that the housing supply increases with the size of the population (an assumption that is often maintained in the related literature). As is specified in more detail below I assume that $\overline{H}_t = \overline{H}_0 e^{n\chi t}$, where $0 \leq \chi \leq 1$ is a parameter that captures the fact that the housing supply might not fully keep pace with population growth.

In the model there are renters and owners. The housing stocks available for renters and owners are denoted by $\overline{H}_t^r$ and $\overline{H}_t^o$, respectively. For the latter I furthermore assume that only a part $\overline{H}_t^{om}$ of the owner-occupied houses are actually on the market while a part $\overline{H}_t^{od}$ is held by direct/dynastic/outright owners that stick to their dwelling, maybe because of sluggishness or out of a sense of family obligations (see on this below). It thus holds that:

$$\overline{H}_t = \overline{H}_t^r + \overline{H}_t^o = \overline{H}_t^r + \overline{H}_t^{om} + \overline{H}_t^{od} = \left(\kappa_H^r + \kappa_H^{om} + \kappa_H^{od}\right) \overline{H}_t, \tag{3}$$

where κ_H^r, κ_H^{om} and κ_H^{od} denote the shares of the total housing stock that are allocated to the three types of dwellers with $\kappa_H^r + \kappa_H^{om} + \kappa_H^{od} = 1$. In the following I describe the three segments of the housing market in more detail.

2.2.1 Rented houses

For each type of dwelling there are two important prices. For the rental properties $\overline{H}_t^r$ the price for housing services P_{st}^r (the "rent") indicates how much a tenant has to pay per unit of housing in order to *use* the housing services for one period. On the other hand, the house price P^r_{ht} states how much an investor has to pay in order to *purchase* one unit of the rental housing stock. Furthermore, it is assumed that the value of the housing stock depreciates at a constant rate δ_h.[4]

The rent and the purchase price are closely related to each other (see Svensson 2023). In particular, the advantage of holding a rental unit is twofold. On the one hand, an investor gets the rent P^r_{st} that is paid for using the unit diminished by the amount $\delta_h P^r_{ht}$ that is needed to hold its service value intact. On the other hand, the investor also benefits from any appreciation in the value of the housing unit, i.e. $\dot{P}^r_{ht} = \frac{dP^r_{ht}}{dt}$. The rate of return r_{ht} on investments into rental housing is thus given by:

$$r_{ht} = \frac{P^r_{st} - \delta_h P^r_{ht} + \dot{P}^r_{ht}}{P^r_{ht}} = \frac{P^r_{st}}{P^r_{ht}} - \delta_h + \frac{\dot{P}^r_{ht}}{P^r_{ht}}.$$

This expression can be solved for the purchasing price P^r_{ht}:

$$P^r_{ht} = \frac{P^r_{st}}{r_{ht} + \delta_h - \frac{\dot{P}^r_{ht}}{P^r_{ht}}}. \tag{4}$$

2.2.2 Owner-occupied houses on the market

For owner-occupiers the situation is somewhat different. The "buying owners", i.e. the ones that have to actually purchase their home (and do not inherit it as a "family property"), face a per unit price of P^o_{ht}. In order to highlight the parallel to the renters it is instructive to assume that the owner-occupiers are completely flexible in their behavior and that they are constantly buying and reselling their homes (abstracting from any transaction costs). Furthermore, it is assumed that these purchases are entirely financed by mortgages with a mortgage interest rate r_{mt}. While occupying their dwelling, households have to pay the maintenance costs[5] while at the same time benefiting from the valuation gains. The user cost of owning (or equivalently: "the imputed rent") is thus given by $P^o_{st} = P^o_{ht}\left(r_{mt} + \delta_h - \frac{\dot{P}^o_{ht}}{P^o_{ht}}\right)$ (see again Svensson 2023). For the owner segment there thus

[4] Alternatively, one could also assume that the depreciation were proportional to the rent P^r_{st} or to aggregate labor income Y_{Lt}. The formulation where depreciation is proportional to the house value is, however, most commonly employed in the related literature, in particular since it is assumed to include also other factors like property taxes. See, e.g., Poterba (1984).

[5] For the sake of simplicity I assume that the maintenance costs are the same for rented and owner-occupied houses.

holds a condition parallel to the rental market expressions (4):

$$P_{ht}^{o} = \frac{P_{st}^{o}}{r_{mt} + \delta_h - \frac{\dot{P}_{ht}^{o}}{P_{ht}^{o}}}. \tag{5}$$

2.2.3 Steady state for the rented and owner-occupied markets

In the steady state house prices grow at the rate $\widetilde{g}$ where:[6]

$$\widetilde{g} \equiv g + n(1-\chi). \tag{6}$$

The rates of return will also be constant in the steady state even though they do not have to be equal (e.g. due to different risk-return profiles). In particular, I assume that:

$$r_{ht} = r_{kt} - \xi_h, r_{mt} = r_{kt} - \xi_m, \tag{7}$$

where ξ_h and ξ_m are risk premia with $\xi_h \geq 0$ and $\xi_m \geq \xi_h$ (such that $r_{kt} \geq r_{ht} \geq r_{mt}$).[7] The steady-state price-to-rent ratios are thus given by:[8]

$$\frac{P_{ht}^{r}}{P_{st}^{r}} = \frac{1}{r_h + \delta_h - \widetilde{g}}, \frac{P_{ht}^{o}}{P_{st}^{o}} = \frac{1}{r_m + \delta_h - \widetilde{g}}. \tag{8}$$

In section 4.2.3 I will show how the equilibrium rent is determined in a simple model and which parameters might have an impact on its size.

2.3 National accounting

In order to calculate easily comparable wealth-to-income ratios it is necessary to first define an income concept that is in line with the conventions concerning net domestic and net national product (Piketty & Zucman 2014, Grossmann, Larin & Steger 2021). This

[6] This follows from the fact that in equilibrium the total value of houses has to grow at rate $n+g$. For the rented segment it thus has to hold that $\frac{dP_{ht}^{r}\overline{H}_t^{r}}{dt}\frac{1}{P_{ht}^{r}\overline{H}_t^{r}} = \frac{\dot{P}_{ht}^{r}}{P_{ht}^{r}} + \frac{\dot{\overline{H}}_t^{r}}{\overline{H}_t^{r}} = n+g$. From $\overline{H}_t = \overline{H}_0 e^{n\chi t}$ it follows that $\frac{\dot{\overline{H}}_t^{r}}{\overline{H}_t^{r}} = \chi n$ and thus $\frac{\dot{P}_{ht}^{r}}{P_{ht}^{r}} = n + g - \chi n = g + n(1-\chi)$. Parallel reasoning also holds for the owner-occupied market with P_{ht}^{o} and $\overline{H}_t^{om}$ (noting that only this part of the owned stock is actually on the market).

[7] For models where the risk premia are derived from explicit assumptions involving stochastic returns, risk aversion and portfolio choices see, e.g., Piazzesi & Schneider (2016).

[8] These relations are well-known from the asset-market approach of the housing market (cf. Poterba 1984).

involves a number of issues ranging from the inclusion of both non-housing and housing goods to the consideration of capital gains and deductions.[9]

In traditional national accounts the gross domestic product is defined as:

$$GDP_t^{NA} = Y_{Nt} + P_{st}^r \overline{H}_t^r + P_{st}^o \overline{H}_t^o \tag{9}$$

where $Y_{Nt} = K_t^\alpha (\mathcal{A}_0 e^{gt} L_t)^{1-\alpha}$ stands for the gross domestic production of normal goods (see (1)) while $P_{st}^r \overline{H}_t^r$ and $P_{st}^o \overline{H}_t^o$ capture the production of housing services in the rented and owner-occupied segments, respectively. This formulation of national income excludes, however, capital gains. As shown, e.g., by Robbins (2018) the neglect of capital gains leads to inconsistencies in the context of theoretical models. I will therefore use the Haig-Simmons definition of national income:

$$GDP_t = Y_t = Y_{Nt} + P_{st}^r \overline{H}_t^r + P_{st}^o \overline{H}_t^o + \dot{P}_{ht}^r \overline{H}_t^r + \dot{P}_{ht}^o \overline{H}_t^o, \tag{10}$$

where $\dot{P}_{ht}^j = \frac{dP_{ht}^j}{dt}$ for $j \in \{r, o\}$. For a discussion of the Haig-Simmons concept see, e.g., Robbins (2018) and Fagereng et al. (2019).

Physical capital K_t and the value of the housing stock depreciate at the rates δ_k and δ_h, respectively. The net domestic product NDP_t is thus given by:[10]

$$NDP_t = GDP_t - \delta_k K_t - \delta_h \left(P_{ht}^r \overline{H}_t^r + P_{ht}^o \overline{H}_t^o \right). \tag{11}$$

2.4 Aggregate asset supply and wealth-to-income ratios

The aggregate asset supply (or equivalently the total demand for wealth) is given by

$$W_t^d = W_{Kt} + W_{Hrt} + W_{Hot} + W_{Dt}, \tag{12}$$

where $W_{Kt} = K_t$, $W_{Hrt} = P_{ht}^r \overline{H}_t^r$, $W_{Hot} = P_{ht}^o \overline{H}_t^o = P_{ht}^o \left(\overline{H}_t^{om} + \overline{H}_t^{od} \right)$ and $W_{Dt} = \mathcal{D}_t$ stands for a possible stock of government bonds $\mathcal{D}_t$. The total amount of housing assets is defined as $W_{Ht} = W_{Hrt} + W_{Hot}$. Alternatively, one can also split the aggregate asset

[9] In this paper I focus on a closed economy and thus abstract from net foreign assets and thus also from the distinction between domestic and national products.

[10] In fact, Robbins (2018) defines capital gains as being "equal to the change in the price of the un-depreciated portion of the asset". This would imply a slightly different definition of net domestic product as $NDP_t = GDP_t - \delta_k K_t - \delta_h P_{ht}^r \overline{H}_t^r (1 + \frac{\dot{P}_{ht}^r}{P_{ht}^r}) - \delta_h P_{ht}^o \overline{H}_t^o (1 + \frac{\dot{P}_{ht}^o}{P_{ht}^o})$. The two formulations will, however, have very similar implications since the product $\delta_h \frac{\dot{P}_{ht}^j}{P_{ht}^j}$ is likely to be small.

supply in the supply of financial (or liquid) assets and the value of owner-occupied assets: $W_t^d = W_{Ft}^d + W_{Ot}^d$ where $W_{Ft}^d = W_{Kt} + W_{Hrt} + W_{Mt} + W_{Dt}$ with $W_{Mt} = M_t$ denoting the value of outstanding mortgages and where $W_{Ot}^d = W_{Hot} - W_{Mt} = P_{ht}^o \overline{H}_t^{od} + \left(P_{ht}^o \overline{H}_t^{om} - M_t\right)$ stands for the net worth of the stock of owner-occupied housing (i.e. its market value minus the value of outstanding mortgage debt).

Total wealth and the various subaggregates can be related to any of the concepts of national income that have been discussed above. The wealth-to-normal-goods ratio, e.g., is defined as:

$$\beta_t^N = \frac{W_t^d}{Y_{Nt}} = (1-\alpha)\frac{W_t^d}{Y_{Lt}}, \tag{13}$$

where I use the fact that $Y_{Lt} = (1-\alpha)Y_{Nt}$. This is a useful concept in the context of the theoretical model since total savings of households will also depend on aggregate labor income Y_{Lt}.

In a similar fashion one can define $\beta_{Kt}^N = \frac{W_{Kt}}{Y_{Nt}}$, $\beta_{Ht}^N = \frac{W_{Ht}}{Y_{Nt}}$, $\beta_{Hrt}^N = \frac{W_{Hrt}}{Y_{Nt}}$, $\beta_{Hot}^N = \frac{W_{Hot}}{Y_{Nt}}$, $\beta_{Dt}^N = \frac{W_{Dt}}{Y_{Nt}}$ and $\beta_{Mt}^N = \frac{W_{Mt}}{Y_{Nt}}$. For later reference one can use equations (2), (4) and (5) to derive:

$$\beta_{Kt}^N = \frac{\alpha}{r_{kt} + \delta_k}, \tag{14}$$

$$\beta_{Hrt}^N = \frac{P_{st}^r \overline{H}_t^r}{Y_{Nt}} \frac{1}{r_{ht} + \delta_h - \widetilde{g}}, \tag{15}$$

$$\beta_{Hot}^N = \frac{P_{st}^o \overline{H}_t^o}{Y_{Nt}} \frac{1}{r_{mt} + \delta_h - \widetilde{g}}. \tag{16}$$

For the two subgroups of owner-occupied houses the ratios are $\beta_{Homt}^N = \frac{P_{st}^o \overline{H}_t^{om}}{Y_{Nt}} \frac{1}{r_{mt}+\delta_h-\widetilde{g}} = \frac{\kappa_H^{om}}{\kappa_H^{om}+\kappa_H^{od}} \beta_{Hot}^N$ and $\beta_{Hodt}^N = \frac{P_{st}^o \overline{H}_t^{od}}{Y_{Nt}} \frac{1}{r_{mt}+\delta_h-\widetilde{g}} = \frac{\kappa_H^{od}}{\kappa_H^{om}+\kappa_H^{od}} \beta_{Hot}^N$.

I assume the the entire supply of financial assets W_{Ft}^d is held by financial funds that operate under the condition of perfect competition. The funds collect all financial savings in the economy, undertake all investments on behalf of the customers (the households) and hand out the returns which constitute the households' asset income. Using the definition of financial wealth from above ($W_{Ft}^d = W_{Kt} + W_{Hrt} + W_{Mt} + W_{Dt}$) the average interest rate is given by:

$$r_t = \frac{K_t}{W_{Ft}^d} r_{kt} + \frac{P_{ht}^r \overline{H}_t^r}{W_{Ft}^d} r_{ht} + \frac{M_t}{W_{Ft}^d} r_{mt} + \frac{D_t}{W_{Ft}^d} r_{dt}, \tag{17}$$

where the interest rates $r_{ht} = r_{kt} - \xi_h$ and $r_{mt} = r_{kt} - \xi_m$ have been defined in equation (7) and where similarly $r_{dt} = r_{kt} - \xi_d$. Note that for the assumption of continuous mortgage-financing the total value of mortgages equals the value of the self-acquired stock, i.e. $M_t = P^o_{ht}\overline{H}^{om}_t$.

The ratio β^N_t (and all other ratios) can be easily transformed into alternative wealth-to-income ratios as discussed in appendix A.1. The related empirical literature, e.g., often divides aggregate wealth by the net domestic product. In this case (which I will define as β_t) one can write:

$$\begin{aligned}\beta_t \equiv \beta^{NDP}_t &= \frac{W^d_t}{NDP_t} = \beta^N_t \frac{Y_{Nt}}{NDP_t} \\ &= \beta^N_t \frac{1}{1 + \frac{P^r_{st}\overline{H}^r_t}{Y_{Nt}} + \frac{P^o_{st}\overline{H}^o_t}{Y_{Nt}} + \frac{\dot{P}^r_{ht}\overline{H}^r_t}{Y_{Nt}} + \frac{\dot{P}^o_{ht}\overline{H}^o_t}{Y_{Nt}} - \frac{\delta_h P^r_{ht}\overline{H}^r_t}{Y_{Nt}} - \frac{\delta_h P^o_{ht}\overline{H}^o_t}{Y_{Nt}} - \frac{\delta_k K_t}{Y_{Nt}}}.\end{aligned}$$

In appendix A.1 I show that in a steady state with $r_{kt} = r_{ht} = r_{mt} = r$ and $\beta^N_D = 0$ the ratio of net domestic product to non-housing output can be written as:

$$\frac{NDP_t}{Y_{Nt}} = 1 + r\beta^N - \alpha = 1 + \alpha\frac{\beta_H}{\beta_K} - \delta_k\beta^N, \tag{18}$$

where $\frac{\beta_H}{\beta_K} = \frac{P^r_{ht}\overline{H}^r_t + P^o_{ht}\overline{H}^o_t}{K_t}$ is the ratio of housing wealth to physical capital wealth. In the absence of housing ($\frac{\beta_H}{\beta_K} = 0$) the net domestic product is always smaller than the non-housing output (with $NDP_t = Y_{Nt}$ for $\delta_k = 0$). This, however, is no longer true for the general situation where it might be the case that the inclusion of housing services exactly counterbalances the subtraction of depreciation such that again $NDP_t = Y_{Nt}$. This will happen if $r\beta^N = \alpha$ which is not an implausible condition (e.g. $\alpha = 0.3$, $\beta^N = 600\%$, $r = 5\%$). For the following benchmark calibrations it will hold that $\frac{NDP_t}{Y_{Nt}}$ is between 95% and 98%.

3 Demand side

The aggregate wealth-to-income ratios derived in the last section depend on the house prices and on the interest rate which are so far undetermined. As a short-cut one could make the assumption that dwellers spend a fixed share of their labor income on housing services together with the assumption of a small-open economy structure with an exogenously given interest rates. In appendix B I study a simple example that is constructed along these lines. For the benchmark case of the model, I want to focus, however, on the case of interest rates that are the endogenous outcome of the intersection between assets supply and asset demand. In order to derive the asset demand (or equivalently the schedule of wealth supply) it is thus necessary to model the savings behavior of the households in more detail. This is sketched in this section where large parts of the derivations are relegated to appendix A.3.

3.1 Demography

The demographic structure of the household side follows Piketty (2011).[11] It is based on a continuous-time OLG model within a deterministic framework. People become adults at age A, are continuously employed until retirement at age R and die at age D. Everybody in this gender-free model has exactly one child at age E and thus everybody will inherit at age $I = D - E$ (if there are bequests). It is assumed that $A \leq I \leq R$.

The notation (following again Piketty 2010) distinguishes between calendar time t, cohort birth year x and age a. The size of the cohort born in period x is thus denoted by $N^x(0)$, while $N^x(a)$ refers to the size of the cohort born in period x at age a (which happens in time $t = x + a$). All members of a cohort are assumed to reach the maximum age D and the cohort size is assumed to grow at rate n:

$$N^x(a) = N^x(0) = N^0 e^{nx}, \tag{19}$$

with the normalization $N^0 = 1$. It follows that $N^{t-a}(a) = N_t(a) = e^{n(t-a)}$. The sizes of the young (working), the old (retired) and the total (adult) populations at time t are given by:

$$N_t^y = \int_A^R N_t(a)\,\mathrm{d}a, N_t^o = \int_R^D N_t(a)\,\mathrm{d}a, N_t = N_t^y + N_t^o. \tag{20}$$

Note that labor supply is identical to the young population, i.e. $L_t = N_t^y$. As stated in appendix A.2 the latter can be calculated as $N_t^y = \frac{e^{-nA} - e^{-nR}}{n} e^{nt}$ which was already used in section 2.1 above (as $L_t = L_0 e^{nt}$). For the assumption that cohort sizes are constant and normalized to 1 it holds that $N_t = D - A$, $N_t^y = R - A$ and $N_t^o = D - R$.

As mentioned in section 2.2, I assume that the housing stock changes with the size of the population. In particular, the housing stock available for cohort x is given by $H^x = H^0 e^{\chi n x}$, where $0 \leq \chi \leq 1$ captures how sensitive the housing stock reacts to

[11] In particular, I follow section 5 in Piketty (2010) (which is the extended working paper version of Piketty (2011)).

population growth. The aggregate housing supply comes out as:[12]

$$\overline{H}_t = \int_A^D H_t(a)\,\mathrm{d}a = H^0 \int_A^D e^{\chi n(t-a)}\,\mathrm{d}a = H^0 \frac{e^{-n\chi A} - e^{-n\chi D}}{n\chi} e^{n\chi t} \tag{21}$$

which was used as $\overline{H}_t = \overline{H}_0 e^{n\chi t}$ above.

3.2 Factor incomes

As stated above aggregate labor income is given by $Y_{Lt} = (1-\alpha)Y_{Nt}$. It holds that aggregate labor income is the total of age-specific labor incomes $Y_{Lt}(a)$, i.e. $Y_{Lt} = \int_A^D Y_{Lt}(a)\,\mathrm{d}a = \int_A^D Y_L^{t-a}(a)\,\mathrm{d}a$. The per adult averages of all aggregate variables are denoted by lower-case variables, i.e. $y_{Nt} = Y_{Nt}/N_t$,$y_{Kt} = Y_{Kt}/N_t$, $y_{Lt} = Y_{Lt}/N_t$, $y_{Lt}(a) = Y_{Lt}(a)/N_t(a)$. Finally, I also assume that households might have different productivities. In particular, it is assumed that each cohort contains a continuum $i \in [0,1]$ of *types* that differ in their labor productivity $y_{Lti}(a)$ with $y_{Lt}(a) = \int_i y_{Lti}(a)\,\mathrm{d}i$.[13] The productivity *growth rates*, however, are assumed to be equal across types and age groups and identical to the rate g.

3.3 Pension system

There exists a pay-as-you-go pension system that is financed by a constant contribution rate τ_ρ and that offers a flat net replacement rate $\rho \leq 1$ to all individual that are older than the retirement age R. As in Piketty (2010) pension income is integrated into labor income $y_{Lti}(a)$ which is thus interpreted as "augmented labor income" which corresponds to net-of-contribution-rate labor income for working adults and to pension income for retired adults. Put formally one can thus write:

$$\begin{aligned} y_{Lti}(a) &= (1-\tau_\rho)\mathrm{y}_{Lti} \text{ for } a \in [A, R[\\ y_{Lti}(a) &= \rho(1-\tau_\rho)\mathrm{y}_{Lti} \text{ for } a \in [R, D], \end{aligned} \tag{22}$$

[12] For the moment I do not model a housing production/construction sector which combines land, labor and normal goods and services in order to produce "dwellings". For a model along these lines see e.g. Grossmann, Larin & Steger (2021).

[13] In fact, in order to take population growth into account it is assumed that each cohort consists of $N^x(a)$ "clones" where each clone itself consists of a continuum $i \in [0,1]$ of types.

where y_{Lti} stands for the pre-contribution-rate labor income of type i in period t. I thus abstract from the existence of seniority wages and all workers of type i get the same wage in a certain period of time independent of their age $a \in [A, R[$.

One can calculate the contribution rate τ_ρ^* that is necessary to have a balanced pay-as-you-go pension system in each period of time. In appendix A.2 I show that this comes out as:

$$\tau_\rho^* \equiv \frac{\rho N_t^o}{N_t^y + \rho N_t^o} = \frac{\rho}{\rho + \frac{e^{n(R-A)}-1}{1-e^{-n(D-R)}}} = \frac{\rho\left(1 - e^{-n(D-R)}\right)}{\rho\left(1 - e^{-n(D-R)}\right) + e^{n(R-A)} - 1}, \tag{23}$$

where I use the expressions for N_t^y and N_t^o given in the appendix. For constant cohort sizes ($n = 0$) this reduces to $\tau_\rho^* = \frac{\rho(D-R)}{R-A+\rho(D-R)}$. In the following it is assumed that $\tau_\rho = \tau_\rho^*$ and thus the pension system is always balanced. When $\rho = 1$ ($\tau_\rho^* = \frac{N_t^o}{N_t}$) then households have a constant (average) income stream over their lifetime.

3.4 Heterogeneous groups

The population is divided into various groups that differ in size, income, dwelling type, and certain preference parameters. In particular, I consider four groups: renters, buying owners with mortgages, outright (or direct) owners and the very wealthy (i.e., the top 1%), abbreviated by the superscripts "r", "om", "oo" and "w", respectively.

- Size: The size of each cohort $N^x(a)$ is partitioned into the four groups with $N^{j,x}(a) = \kappa_N^j N^x(a)$ where κ_N^j stands for the share of group $j \in \{r, om, oo, w\}$ with $\sum_j \kappa_N^j = 1$. The share of the top 1% is fixed per definition at $\kappa_N^w = 0.01$. The composition of the remaining three groups is treated as given (where it is implicitly assumed that their relative sizes are the outcome of individual preferences, cultural norms, financial constraints and regulations etc.). The groups of renters and owners with mortgages are in control of fractions κ_H^r and κ_H^{om} of the total housing stock $\overline{H}_t$, respectively (see section 2.2). In as far as the directly owned housing stock is concerned, I assume that also the top 1% fall into this group. It thus holds that $\overline{H}_t^{od} = \overline{H}_t^{oo} + \overline{H}_t^{w} = (\kappa_H^{oo} + \kappa_H^{w})\,\overline{H}_t$.

- Income: I assume that $y_{Lti} = d_y^j y_{Lt}$ if i is member of group j. In other words y_{Lti} can only take on four different values and it holds that $\sum_j \kappa_N^j d_y^j = 1$. In fact, for the sake of simplicity I will assume that only the wealthy have a different income level and that $d_y^r = d_y^{om} = d_y^{oo} = d_y$. For a given d_y^w it thus follows that $d_y = \frac{1-d_y^w \kappa_N^w}{1-\kappa_N^w}$.

- Preferences: As will be discussed in the next section, the four groups also differ in the utility they get from housing services (higher for owners) and in the strength of the bequest motive (higher for the top 1%).

3.5 Utility functions

Households have to take three decisions: the intratemporal choice how to allocate a given level of expenditures between housing and non-housing consumption; the intertemporal choice how to set expenditure levels across time; and the intergenerational choice about leaving a potential bequest. I analyze these decisions in the benchmark framework of a warm-glow model.[14] For the sake of readability I will in the following leave out the group-specific superscript "j" whenever it does not give rise to misunderstandings. I will thus write, e.g., $h_i^x(a)$ instead of $h_i^{j,x}(a)$, y_{Li}^x instead of $y_L^{j,x}$ etc. I will keep the superscript, however, for parameters that represent specific group characteristics like the dwelling-specific utility parameter η^j and the group-specific bequest motive s_B^j.

3.5.1 The intratemporal choice between housing and non-housing consumption

Households have to decide how a given expenditure level $\varepsilon_i^x(a)$ is allocated between non-housing consumption $c_i^x(a)$ and housing services $h_i^x(a)$. For the intratemporal utility function I use the following specification:

$$u(c_i^x(a), h_i^x(a)) \quad = \quad \frac{\left(\eta^j h_i^x(a)\right)^{\gamma} \left(c_i^x(a)\right)^{1-\gamma}}{(\gamma)^{\gamma}(1-\gamma)^{1-\gamma}}$$

where γ measures the relative weight of housing and η^j captures the fact that renters and owners might experience different levels of utility from their dwelling, even if the housing is identical in every other respect like size, amenities etc. In particular, it is assumed that $\eta^o \geq \eta^r$.[15]

[14] A previous version of the paper also included the results of a model with exogenously given savings rates and with accidental bequests. These are available upon request.

[15] The utility mark-up of owning might have to do with the fact that people value the sense of security, the possibility to modify and to adapt the buildings, the function of homeownership as a security for catastrophic risk etc. It might also be a short-cut that captures possible tax advantages (or maybe disadvantages) of owning vs. renting and thus people would ceteris paribus prefer to own rather than to rent their shelter. Similar formulation are also used by, e.g., Iacoviello & Pavan (2013) or Kaplan et al. (2020).

Renters: The period budget constraint of renters is given by $c_i^x(a)+P_s^x(a)h_i^x(a) = \varepsilon_i^x(a)$ (where the price here stands for $P_s^{r,x}(a)$). Maximizing the utility function with respect to this budget constraint leads to the optimal levels:

$$\begin{aligned} c_i^x(a) &= (1-\gamma)\varepsilon_i^x(a) \\ h_i^x(a) &= \gamma \frac{\varepsilon_i^x(a)}{P_s^x(a)}. \end{aligned} \tag{24}$$

This set-up thus implies that households might change their housing consumption continuously (depending on the development of $\varepsilon_i^x(a)$ and $P_s^x(a)$). This assumption of a perfectly flexible housing demand is certainly rather implausible and is mainly done for convenience (to keep the intra- and intertemporal decisions separated). It will turn out, however, that in the steady state of the benchmark model the housing demand stays in fact constant over the lifecycle.

Owners with mortgages: In order to simplify the exposition and to facilitate the comparison to the group of renters I also assume (as already mentioned in section 2.2) that the normal owner-occupiers are completely flexible and that they are constantly buying and reselling their homes without transaction costs. These purchases are entirely financed by mortgages at the mortgage interest rate r_{mt}. As discussed in section 2.2, this means that the imputed rent of these owners is given by $P_{st}^o = P_{ht}^o \left(r_{mt} + \delta_h - \frac{\dot{P}_{ht}^o}{P_{ht}^o} \right)$. Returning to the cohort-perspective this means that the period budget constraint is still given by $c_i^x(a) + P_s^x(a)h_i^x(a) = \varepsilon_i^x(a)$ only that the price now stands for $P_s^{o,x}(a)$. The optimal choices are thus still given by (24).

Outright owners: For direct owners the situation is different. They regard the house as a family property which is passed on from generation to generation. In particular, I assume that they inherit the house at birth (e.g. from their grandparents) and they just pay the maintenance costs until their death when they pass it on to their own grandchildren.[16] Thus they do not have an intratemporal choice and they simply set $c_i^x(a) = \varepsilon_i^x(a) - \delta_h P_h^x(a)h_i^x(a)$ for a house of given (inherited) size $h_i^x(a) = h_i^x(A)$.

This structure is of course highly stylized, but it allows to consider the empirically

[16] In fact, in the presence of population growth and with $\chi > 0$ one has to assume that in addition to the bequest of the grandparents also part of the newly created housing stock is transformed into this dynastic category. For a more detailed account one could also model the transition between the different categories of dwellings and dwellers. This, however, is beyond the scope of the present paper and is left for future research.

important group of outright owners in a straightforward and tractable manner. The stylized assumption of completely passive outright owners who simply use and pass on their inherited home is thereby meant to capture three observable phenomena. First, a certain proportion of real estate is under the control of real estate trusts (fee tails, entails, fideicommis etc.) that prohibits (or at least considerably restricts) heirs to sell the inherited land. Second, a certain percentage of the population does not seem to be *willing* to trade their home even if they were *able* to do so, for example because they feel an obligation to a generation-old "family house" etc. Third, and probably most relevant, in the real world the two groups of owners with mortgages and outright owners are not separate entities but many households rather assume these roles in sequence. They start of as mortgage-holders and turn into outright owners after having paid of their debts from when on they simply stick to their home until death ("aging in place") without viewing it as a manageable asset. The assumed structure is meant to capture this constellation in a tractable manner.

3.5.2 Intertemporal utility

For the benchmark specification I assume that the second and third stage are determined by a warm-glow model similar to the version used in Piketty (2011).[17] What is more, I will focus on a specific variant of the model where households do not reckon with the receipt of a bequest (or—equivalently—they cannot borrow against it.)

The intertemporal utility function is assumed to have the following form:

$$U_i^x(A) = \left\{ \frac{\int_A^D e^{-\theta(a-A)} \left(u_i^x(a)\right)^{1-\sigma} \, \mathrm{d}a}{\int_A^D e^{-\theta(a-A)} \, \mathrm{d}a} \right\}^{\frac{1}{1-\sigma}}, \tag{25}$$

where θ stands for the rate of time preference, $\frac{1}{\sigma} > 0$ for the intertemporal elasticity of substitution and where the intratemporal utility $u_i^x(a)$ depends on $c_i^x(a)$ and $h_i^x(a)$ as shown in (24).[18]

[17] In fact, Piketty (2011) calls his model a "wealth-in-the-utility model". Since in more recent years this expression increasingly has been reserved for a slightly different class of model, I choose here the name "warm-glow".

[18] This specification implies that U measures the *average* period utility weighted with the discount factor $e^{-\theta(a-A)}$. This is irrelevant for the optimization but makes more sense for the intergenerational context and is also necessary for the limiting case of $\sigma \to 1$.

3.5.3 Intergenerational utility

Households who are only guided by utility function (25) will exclusively save for live cycle reasons, i.e., in order to smooth consumption over time. There exist, however, also a number of other, arguably equally important savings motives. A crucial motive, e.g., is people's intention to leave a bequest to their survivors. In order to consider this motive I extend the basic framework in a straightforward manner by using the warm-glow specification introduced by Piketty (2010). In particular, I assume that each household maximizes an intergenerational utility function of the form:

$$V_i^x(A) = (1 - s_B^j) \log \left(U_i^x(A) \right) + s_B^j \log \left(e^{-\theta(D-A)} w_i^x(D) \right). \tag{26}$$

Overall utility thus depends on the average utility of own (housing and non-housing) consumption $U_i^x(A)$ (as defined in (25)) and on level of wealth $w_i^x(D)$ that is available at the moment of death at age D. Wealth might consist of financial wealth $w_{Fi}^x(D)$ and also of the value of an owner-occupied house $w_{Hi}^x(D)$ which is still in the possession of the deceased. The parameter s_B^j stands for the strength of the bequest motive and it might be group-specific. As noted by Piketty (2010) one could also give other reasons for the relevance of end-of-life wealth for overall utility. It might, e.g., capture a precautionary savings motive (in the presence of uninsurable shocks to income or health) or it might be related to direct utility from social status or power that goes hand-in-hand with a high level of wealth. For the sake of simplicity I again assume (as for the income inequality parameter d_y^j) that the strength of this motive is the same for the bottom 99%, i.e. $s_B^r = s_B^{om} = s_B^{oo} = s_B$.

4 Solution

4.1 Household values

As in the previous section I will discuss the solution of the warm-glow model without explicitly differentiating between the four groups and continue to use the superscript "x" instead of "j, x". This is possible since due to a number of assumptions the optimal choice is basically identical for all four groups. I will only turn to the heterogeneities when there are important exceptions.

The solution to the model can be derived in a number of steps that are spelled out in appendix A.3. The intertemporal and intergenerational utility functions determine the path of expenditures which are given by:

$$\varepsilon_i^x(a) = \begin{cases} (1-\tau)\mathrm{y}_{Li}^x(A)(1-s_L^j)e^{g_\varepsilon(a-A)} & \text{for } a \in [A, I[\\ (1-\tau)\mathrm{y}_{Li}^x(A)(1-s_L^j)e^{g_\varepsilon(a-A)} + (1-\breve{s}_B^j)b_{Fi}^x e^{g_\varepsilon(a-I)} & \text{for } a \in [I, D] \end{cases} \tag{27}$$

where

$$g_\varepsilon = \frac{r-\theta}{\sigma} + \gamma\frac{\sigma-1}{\sigma}\widetilde{g} \tag{28}$$

is the optimally chosen expenditure growth rate while the coefficients s_L^j and $\breve{s}_B^j$ denote the savings rates out of net labor income $(1-\tau)\mathrm{y}_{Li}^x(A)$ and out of the bequest b_{Fi}^x, respectively. Their explicit formulas are given in appendix A.3.[19] Note that the interest rate r in expression (28) refers to the average interest rate that the households receive from the financial funds that aggregate all available financial assets of the economy (see equation (17)).

In the next step one can derive the equilibrium value for the financial bequest b_{Fi}^x. This is determined by two equations. On the one hand by the intergenerational utility function that implies that households plan to bequest a share s_B^j of their lifetime resources and their terminal wealth level $w_{Fi}^x(D)$ will thus be a function of their lifetime labor income $\widetilde{y}_{Li}^x = (1-\tau_y)\int_A^D y_{Li}^x(a)e^{-r(a-A)}\,\mathrm{d}a$ and of the bequest b_{Fi}^x they have received themselves. On the other hand, along a stationary equilibrium this end-of-life wealth $w_{Fi}^x(D)$ has to correspond to the per descendant bequest level of the cohort born in period $x+(D-I)$, i.e.:

$$w_{Fi}^x(D) = b_{Fi}^{x+(D-I)}\frac{N^{x+(D-I)}}{N^x} = b_{Fi}^{x+(D-I)}e^{n(D-I)}. \tag{29}$$

Equating the last two equations one can thus derive (for details see appendix A.3.2) the equilibrium bequest level as:

$$b_{Fi}^x = s_B^j\frac{e^{(r-g)(D-A)}e^{g(I-A)}}{e^{n(D-I)} - s_B^j e^{(r-g)(D-I)}}\widetilde{y}_{Li}^x. \tag{30}$$

In a final step one can then use the expenditure levels $\varepsilon_i^x(a)$ in (27) together with the

[19] For the benchmark calibration I will assume that the preference parameters θ and σ adjust such that the expenditure growth rate follows the productivity growth rate, i.e. $g_\varepsilon = g$. This is also suggested by Piketty (2010) and there exist a number of reasons why this seems like a reasonable assumption. Furthermore, for the benchmark calibration I will also assume that $\chi = 1$ such that $g_\varepsilon = \widetilde{g} = g$. In fact, for the case of outright owners equation (27) only represents a dynamic optimum if this condition $g_\varepsilon = \widetilde{g} = g$ is fulfilled. For numerical examples that deviate from this condition I will assume, however, that also the outright owners determine their path of expenditures $\varepsilon_i^x(a)$ according to equation (27).

bequest level b^x_{Fi} in (30) to derive an explicit expression for the path of wealth $w^x_{Fi}(a) = w^x_{Li}(a) + w^x_{Bi}(a)$ over the lifecycle $a \in [A, D]$ of household i where $w^x_{Li}(a)$ and $w^x_{Bi}(a)$ refer to the labor-income and bequest-related parts of households financial wealth $w^x_{Fi}(a)$.

4.2 Aggregate values

4.2.1 Aggregate household wealth

In order to derive the aggregate values, one has again to explicitly deal with the four heterogeneous groups. The aggregate supply of financial wealth (or the aggregate demand for financial assets) is defined as $W^s_{Ft} = \sum_j W^{j,s}_{Ft}$ where $W^{j,s}_{Ft} = \int_A^D w^j_{Ft}(a) N^j_t(a)\, da$ with equivalent definitions for the labor-income and bequest-related parts of wealth $W^{j,s}_{Lt}$, $W^{j,s}_{Bt}$, respectively. In appendix A.3 I show how one can use various transformation to $w^j_{Lt}(a)$ and $w^j_{Bt}(a)$ to solve for the equilibrium level of aggregate wealth supply in closed form. In particular, using the definitions $\widetilde{\beta}^{j,N}_{Ft} = \frac{W^{j,s}_{Ft}}{Y_{Nt}}$, $\widetilde{\beta}^{j,N}_{Lt} = \frac{W^{j,s}_{Lt}}{Y_{Nt}}$ and $\widetilde{\beta}^{j,N}_{Bt} = \frac{W^{j,s}_{Bt}}{Y_{Nt}}$ the steady state wealth supply schedule can be expressed as:

$$\widetilde{\beta}^{j,N}_F = \widetilde{\beta}^{j,N}_L + \widetilde{\beta}^{j,N}_B. \tag{31}$$

The entire (rather lengthy) expressions for $\widetilde{\beta}^{j,N}_L$ and $\widetilde{\beta}^{j,N}_B$ are stated in equations (83) and (84) in appendix A.3. The schedule of aggregate financial wealth supply is then given by:[20]

$$\widetilde{\beta}^N_F = \sum_j \widetilde{\beta}^{j,N}_F = \widetilde{\beta}^{r,N}_F + \widetilde{\beta}^{om,N}_F + \widetilde{\beta}^{oo,N}_F + \widetilde{\beta}^{w,N}_F. \tag{32}$$

Note that total household wealth also includes the value of the inherited houses of the direct owners, i.e. $\widetilde{\beta}^N = \widetilde{\beta}^N_F + \widetilde{\beta}^N_H = \widetilde{\beta}^N_F + \beta^{oo,N}_{Ho} + \beta^{w,N}_{Ho}$. The value $\widetilde{\beta}^N_H$ is relevant when evaluating total household wealth, but it does not play a role for the determination of the equilibrium interest rate r.[21]

[20] The expression for $\widetilde{\beta}^N_F$ can be simplified for some special assumptions. In the absence of population growth ($n = 0$), of a PAYG pension system ($\rho = 0$), of income taxes ($\tau_y = 0$) and of a bequest motive ($s^j_B = 0$) and assuming in addition that the interest rate equals the growth rate ($r = g$ and $g_\varepsilon = g$) the model reduces to the structure of the "consumption loan economy" of Modigliani (1986). In this case the pattern of wealth follows the famous "triangle" and the wealth-to-income-ratio reduces to $\widetilde{\beta}^N_F = (1-\alpha)\frac{D-R}{2}$.

[21] A different way to see this is that when equating total asset supply and total asset demand the term for the value of the directly owned housing stock simply cancels on both sides of the equation.

4.2.2 Aggregate savings

One can also use the model to derive additional aggregate values that are informative to compare to the empirical counterparts. Aggregate savings are defined as the difference between total income $Y_t = GDP_t$ and total expenditures $\mathcal{E}_t$, i.e.:

$$S_t = Y_t - \mathcal{E}_t \tag{33}$$

where $\mathcal{E}_t = \sum_j \mathcal{E}_t^j$ and $\mathcal{E}_t^j = \int_A^D \varepsilon_t^j(a) N_t^j(a)\,\mathrm{d}a$. For the model of section 4 aggregate expenditures for group j can be calculated as (see appendix A.6):

$$\begin{aligned} \mathcal{E}_t^j &= \kappa_N^j d_y^j Y_{Nt}(1-\alpha)(1-\tau)\frac{e^{-n(D-R)}n}{e^{n(R-A)}-1} \times \\ &\quad \frac{(1-s_L^j)\left(e^{n(D-A)} - e^{(g_\varepsilon - g)(D-A)}\right) + (1-\widetilde{s}_B^j)\left(e^{n(D-I)} - e^{(g_\varepsilon - g)(D-I)}\right)}{n+g-g_\varepsilon}, \end{aligned} \tag{34}$$

where $(1-s_L^j)$ and $(1-\widetilde{s}_B^j)$ are defined in appendix A.3.

Using aggregate savings S_t, the gross savings rate is defined as:

$$\overline{s} = \frac{S_t}{GDP_t}. \tag{35}$$

Since these definitions use the Haig-Simmons income concept, the measure of savings S_t thus includes also all savings from capital gains $\dot{P}_{ht}^r \overline{H}_t^r + \dot{P}_{ht}^o \overline{H}_t^o$.

Empirical measures of the savings rate often use different income concepts by excluding certain income components. In appendix A.6 I show, e.g., that the net-savings rate $\overline{s}^{net}$, e.g., is given by:

$$\overline{s}^{net} = 1 - (1-\overline{s})\frac{GDP_t}{NDP_t}. \tag{36}$$

If depreciations amount to 10% of GDP then a gross savings rate of 30% (25%) corresponds to a net savings rate of 22% (17%). For an overall depreciation rate of 15% (which is closer to the data for OECD countries) the corresponding values are 18% (12%) for gross savings rate of 30% (25%).

Published savings rates typically exclude capital gains both in the numerator (the definition of savings) and the denominator (the definition of national income). One can thus also define a "national-account gross savings rate" $\overline{s}^{NA}$ based on the traditional income concept GDP_t^{NA} from equation (9) as:

$$\overline{s}^{NA} = 1 - \frac{GDP_t}{GDP_t^{NA}} (1 - \overline{s}) . \tag{37}$$

4.2.3 Equilibrium house prices

The equilibrium house price for the rented market is determined by group "r" (the renters) while the price for the owner-occupied market depends on the behavior of group "om" (the owners with a mortgage). The housing demand of the outright owners is pre-determined and they do not participate in the market. Average age-specific housing demand for segments $j \in \{r, om\}$ is given by $h_t^j(a) = \int_i h_{ti}^j(a)\, \mathrm{d}i$ and aggregate housing demand by $\widetilde{H}_t^j = \int_A^D h_t^j(a) N_t^j(a)\, \mathrm{d}a$. The age-specific levels of housing demand have been derived in section 3.5.1 as $h_t^r(a) = \gamma \frac{\varepsilon_t(a)}{P_{st}^r}$ and $h_t^{om}(a) = \gamma \frac{\varepsilon_t(a)}{P_{st}^o}$. Therefore aggregate housing demand can be written as $\widetilde{H}_t^r = \frac{\gamma}{P_{st}^r} \int_A^D \varepsilon_t^r(a) N_t^r(a)\, \mathrm{d}a = \frac{\gamma}{P_{st}^r} \mathcal{E}_t^r$ and $\widetilde{H}_t^{om} = \frac{\gamma}{P_{st}^o} \int_A^D \varepsilon_t^{om}(a) N_t^{om}(a)\, \mathrm{d}a = \frac{\gamma}{P_{st}^o} \mathcal{E}_t^{om}$. The housing market is in equilibrium when aggregate demand equals aggregate supply (see (21)), i.e. $\widetilde{H}_t^r = \overline{H}_t^r$ and $\widetilde{H}_t^{om} = \overline{H}_t^{om}$. The equilibrium rents thus comes out as:

$$P_{st}^r = \frac{\gamma}{\overline{H}_t^r} \mathcal{E}_t^r , P_{st}^o = \frac{\gamma}{\overline{H}_t^{om}} \mathcal{E}_t^{om} .$$

One can use the expression for $\mathcal{E}_t^j$ in (34) to write the house prices in terms of the parameters of the model. The resulting formulas are shown in appendix A.6. I also show there that due to the assumptions that $d_y^r = d_y^{om} = d_y$ and $s_B^r = s_B^{om} = s_B$ it holds that $\frac{\mathcal{E}_t^r}{\mathcal{E}_t^{om}} = \frac{\kappa_N^r}{\kappa_N^{om}}$ and thus:

$$\frac{P_{st}^r}{P_{st}^o} = \frac{\kappa_N^r}{\kappa_N^{om}} \frac{\overline{H}_t^{om}}{\overline{H}_t^r} = \frac{\kappa_N^r}{\kappa_N^{om}} \frac{\kappa_H^{om}}{\kappa_H^r} , \tag{38}$$

where I use the definitions $\overline{H}_t^r = \kappa_H^r \overline{H}_t$ and $\overline{H}_t^{om} = \kappa_H^{om} \overline{H}_t$ from (3). It is reasonable to assume that in the long-run the supply of the rented and the owner-occupied housing stock will adjust such that households are indifferent between renting and owning. One can insert the optimal solutions into the utility functions and show that lifetime utility for renters and owner-occupiers will be the same for $\frac{P_{st}^r}{\eta^r} = \frac{P_{st}^o}{\eta^o}$ or:

$$\frac{P_{st}^r}{P_{st}^o} = \frac{\eta^r}{\eta^o} . \tag{39}$$

Equations (38), (39) and $(\kappa_H^r + \kappa_H^{om} + \kappa_H^{od}) = 1$ can be used to calculate the equilibrium share of the supply for rented and owner-occupied housing for a given share $\kappa_H^{od} = \kappa_H^{oo} + \kappa_H^w$ of directly owned houses:

$$\begin{aligned}
\kappa_H^r &= \frac{\kappa_N^r \eta^o}{\kappa_N^r \eta^o + \kappa_N^{om} \eta^r} \left(1 - (\kappa_H^{oo} + \kappa_H^w)\right), \\
\kappa_H^{om} &= \frac{\kappa_N^{om} \eta^r}{\kappa_N^r \eta^o + \kappa_N^{om} \eta^r} \left(1 - (\kappa_H^{oo} + \kappa_H^w)\right).
\end{aligned} \tag{40}$$

4.2.4 Equilibrium interest rates

One can now proceed to derive the equilibrium interest rate of the economy. It is the interest r^* that clears the market for financial assets. The supply of financial assets (or equivalently the schedule for wealth demand) has been specified in section 2.4 as $W_{Ft}^d = W_{Kt} + W_{Hrt} + W_{Mt} + W_{Dt}$ while the demand for financial assets (or equivalently the schedule for wealth supply) was derived above as $W_{Ft}^s = W_{Lt}^s + W_{Bt}^s$. Dividing these expressions by Y_{Nt} one can also express them in terms of wealth-to-income ratios as:

$$\beta_F^N = \beta_K^N + \beta_{Hr}^N + \beta_M^N + \beta_D^N = \widetilde{\beta}_L^N + \widetilde{\beta}_B^N = \widetilde{\beta}_F^N. \tag{41}$$

The formulas for $\beta_K^N = \frac{\alpha}{r_k + \delta_k}$, $\beta_{Hr}^N = \frac{P_{st}^r \overline{H}_t^r}{Y_{Nt}} \frac{1}{r_h + \delta_h - \widetilde{g}}$ and $\beta_M^N = \frac{P_{st}^o \overline{H}_t^{om}}{Y_{Nt}} \frac{1}{r_m + \delta_h - \widetilde{g}}$ have already been stated in section 2.4. One can now use the equilibrium rents P_{st}^r and P_{st}^o (see (88) and (89)) to derive an expression that only depends on the interest rates r_h and r_m. As far as the wealth-ratio of government bonds β_D^N is concerned I simply assume that it stays constant over time with the tax income rate τ_y adjusting such as to balance the interest payments.[22] The expressions for the wealth-supply ratios $\widetilde{\beta}_L^N$ and $\widetilde{\beta}_B^N$ (see appendix A.3) only depend on the average interest rate r which is provided by the financial funds that hold all financial assets of the economy. The equilibrium of the model is thus given by the solution to equation (41) together with the definition of the average interest rate r in equation (17). These two equations can be solved for the equilibrium levels of r_k and r while the other interest rates are then given by $r_h = r_k - \xi_h$, $r_m = r_k - \xi_m$ and $r_d = r_k - \xi_d$ (see (7)).

[22]In particular, the dynamic equation for government debt is given by $\dot{\mathcal{D}}_t = r_d \mathcal{D}_t - \tau_y Y_{Lt}$. Using the relation that $\mathcal{D}_t = \frac{\beta_D^N}{1-\alpha} Y_{Lt}$ and the assumption that $\dot{\beta}_D^N = 0$ the equation can be rewritten as $\dot{Y}_{Lt} \frac{\beta_D^N}{1-\alpha} = r_d \frac{\beta_D^N}{1-\alpha} Y_{Lt} - \tau_y Y_{Lt}$. Noting that $\frac{\dot{Y}_{Lt}}{Y_{Lt}} = g + n$ this implies that $\tau_y = \frac{\beta_D^N}{1-\alpha}(r_d - (g+n))$.

5 Numerical results

The problem stated in (41) cannot be solved in closed form for the general case and in the following I resort to numerical solutions of calibrated models.

I follow Summers & Rachel (2019) and Platzer & Peruffo (2022) and look at steady-state comparisons between an "initial period" that corresponds to the period 1975-1980 before the decline in real interest rates has started and a "current period" that is meant to reflect the time around 2018 before the outbreak of the recent "polycrisis".

5.1 Calibration

The parameter values are based on data from the World Bank and the OECD referring to the group of advanced countries. In particular, for the initial period I use the following calibration: $g = 3\%$, $n = 1\%$, $A = 20$, $R = 65$, $D = 75$, $\rho = 70\%$ and $\beta_D^N = 20\%$. The parameters related to the production side are set to the standard values $\alpha = 0.33$, $\delta_k = 10\%$, $\gamma = 0.17$ and $\delta_h = 2.5\%$. Furthermore, I assume that government debt amounts to 20% of normal output ($\beta_D^N = 20\%$). In appendix C the data source and the rationale behind this calibration is discussed in more detail. As far as the intertemporal preference parameters θ and σ are concerned I follow the example of Piketty (2010) and assume that they are determined in such a way that the expenditure growth rate follows the productivity growth rate, i.e. $g_\varepsilon = g$. As argued by Piketty (2010) there exist theoretical and empirical reasons why this is a reasonable assumption. In appendix D I report how the results change if one deviates from this assumption (e.g. by using a standard calibration with $\theta = 0.02$ and $\sigma = 2$).

The benchmark model is based on four groups of households as described above. Below I will show that this model is capable to match a number of stylized facts which is not true for models that are based on a less-detailed structure of dwellers. For the sake of comparisons, however, I will often also report the results for different assumptions, e.g. for a three group model (without outright owners, $\kappa_N^{oo} = 0$), a two group model (without owners, $\kappa_N^{om} = \kappa_N^{oo} = 0$) or the model without housing ($\gamma = 0$). The heterogeneous structure can be used to capture and calibrate a number of distributional aspects of the the real-world data. First, I assume that the labor income of the top 1% is higher than the one of the bottom 99% ($d_y^w = 3$). Second, the bequest motives of the bottom 99% (s_B) and the top 1% (s_B^w) can be determined in a way such as to target three empirically observed magnitudes—the wealth-to-income ratio β, the share of the wealth of the top 1%

and the average inheritance flow b_y (for the definition of b_y see appendix A.6). I decided to target a value of $\beta = 350\%$ and a top 1% wealth share of 28% while the inheritance flow b_y was left to be determined by the calibrated model. The choice for the top 1% share follows from the data reported in Alvaredo et al. (2018) while the target values for β follow from data in Piketty & Zucman (2014) and Bauluz et al. (2022) which will be discussed in the next section. Altogether, this leads to a parameter choice of $s_B = 0.0089$ and $s_B^w = 0.093$.[23]

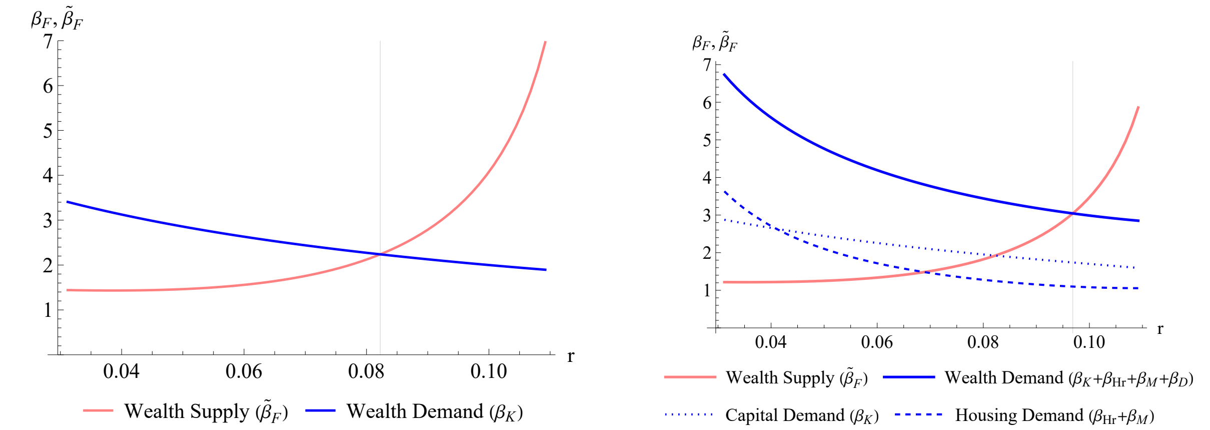
Finally, one has to find a reasonable calibration for the size of the four groups of dwellers and the relative share of the housing stock that is under their control. I assume that in the initial situation there is an equal share of renters and owners where for the latter category there is itself an equal split in self-buying owners and outright owners (i.e. $\kappa_N^r = 50\%$, $\kappa_N^{om} = 25\%$, $\kappa_N^{oo} = 24\%$, $\kappa_N^w = 1\%$).[24] The size of the housing stock available for renters (κ_H^r) and self-buying owners with mortgages (κ_H^{om}) adjusts according to expression (40) such as to make households indifferent between the two types of dwellings. For these calculations I assume that the utility from owner-occupied houses is 20% higher than for rented houses ($\eta^r = 1$, $\eta^o = 1.2$). For the stock of outrightly owned housing I assume that they are equal to the population size in the initial situation ($\kappa_H^{oo} = 24\%$, $\kappa_H^w = 1\%$). Finally, for the risk discounts I assume $\xi_d = 5\%$, $\xi_m = 2\%$ and $\xi_h = 0\%$ (see appendix C for the rationale behind this calibration).

5.2 Steady-state comparisons

The equilibrium interest rate is given by the value that solves equation (41), i.e. that equilibrates wealth supply by households and wealth demand by investors. This is illustrated in Figure (1)—in panel (a) for the commonly used case without housing and in panel (b) for the novel case with housing. For the sake of clarity, I abstract from risk discounts in this picture ($\xi_h = \xi_m = \xi_d = 0$). As can be seen, wealth supply β is increasing in r while both components of wealth demand—capital demand and housing demand—are decreasing in r (at least over the relevant range).

[23] I thus assume that individuals only differ in their intergenerational bequest motive. Alternatively one could also assume that individuals have nonhomothetic preferences that lead to higher savings rates for households with higher permanent income or higher wealth (see e.g. Platzer & Peruffo 2022). Note, however, that in my framework a higher value for the bequest motive s_B^j will also be reflected in higher savings rate out of labor income s_L^j and out of the received bequest $\check{s}_B^j$ (see equations (68) and (75) in appendix A.3).

[24] Data on this issue that cover differ countries and different time periods are sparse. In appendix C.2 I discuss the available evidence and how it was used to inform the calibration of the model.



(a) No housing (b) With housing

Figure 1: The figures shows the supply of financial wealth $\widetilde{\beta}_F$ and the demand for financial wealth β_F which is β_K in panel (a) and $\beta_K + \beta_{Hr} + \beta_M + \beta_D$ in panel (b). For both figures: $g = 3\%$, $n = 1\%$, $A = 20$, $R = 65$, $D = 75$, $\rho = 70\%$, $\alpha = 0.33$, $\delta_k = 10\%$, $s_B = 0.0089$ and $s_B^w = 0.093$, $d_y^w = 3$, $\beta_D^N = 20\%$, $\xi_d = \xi_h = \xi_m = 0\%$ and in addition (for panel b) $\delta_h = 2.5\%$, $\gamma = 0.17$, $\eta^o = 1.2$ and κ_N^j and κ_H^j as stated in the text. The equilibrium is: $r = 8.2\%$, $\beta = 248\%$ (for $\gamma = 0$) and $r = 9.7\%$, $\beta = 346\%$ (for $\gamma = 0.17$).

The equilibrium values for the two cases (now again with $\xi_m = 2\%$ and $\xi_d = 5\%$) come out as:

$$\textbf{With housing:} \quad r = 9.64\%, \beta = 350\%, \beta_H/\beta = 46\%, \tag{42}$$

$$\textbf{Without housing:} \quad r = 8.43\%, \beta = 238\%, \beta_H/\beta = 0\%. \tag{43}$$

The existence of housing provides another investment vehicle for the supply of savings by households. This leads to lower investments into the capital stock and drives up the equilibrium interest rate from 8.4% to 9.6%. On the other hand, however, housing wealth turns out to be an important asset with a portfolio share of $\beta_H/\beta = 46\%$ such that total wealth is still higher for the case with $\gamma = 0.17$ ($\beta = 350\%$) than for the case with $\gamma = 0$ ($\beta = 238\%$).

In a next step one can look at the effect of changes in the crucial parameters. In particular, these parameters are now set to values that reflect the average conditions in the group of advanced countries around 2018. Using again data from the World Bank and the OECD I now take the following calibration: $g = 1.8\%$, $n = 0.5\%$, $D = 82$, $R = 63$ and

$\rho = 60\%$. For government debt I follow Summers & Rachel (2019) and assume an increase from $\beta_D^N = 20\%$ to $\beta_D^N = 70\%$. As far as the parameter d_y^w is concerned, Garbinti et al. (2020) report a constant development of the labor income share of the top 1% in France but an increase by about 50% for total income and I thus use $d_y^w = 4.5$. The bequest motive of the top 1% is assumed to increase to $s_B^w = 0.207$ such that the share of the top 1% in total wealth increase from 28% to 35% as reported in Alvaredo et al. (2018). For the share of dwellers and dwellings I assume that the share of renters (mortgage-owners) decreases (increases) by 10 pp while the share of outright owners stays constant (i.e. $\kappa_N^r = 40\%$, $\kappa_N^{om} = 35\%$, $\kappa_N^{oo} = 24\%$, $\kappa_N^w = 1\%$). I assume, however, that the *stock* of housing controlled by the outright owners increases by 65% for the current situation ($\kappa_H^{oo} = 40\%$, $\kappa_H^w = 1.65\%$). The latter development is in particular meant to capture the increasing size of baby-boomer households who have paid of their mortgages and are now outrightly owning a larger share of the existing housing stock. For a discussion of the rationale behind this calibration see appendix C.2. Finally, the risk discount of mortgages is assumed to increase to $\xi_m = 3\%$.

Table 1 shows the comparison of the initial and the current steady state for the benchmark model. The model implies a decrease in the neutral interest rate by almost 4 pp (from 9.6% to 5.7%) and an increase in the wealth-to-income ratio by almost 250 pp (from 350% to 599%). These results are quite well aligned with the empirical data. In particular, starting with the interest rate levels one can note that Jordà et al. (2019) report a global (weighted) average real return on risky assets (equity and housing) of 7.88% (1950-1980) and 6.66% (post-1980). This is somewhat lower than the model-based average interest rate of 7.7% (taking the average of "initial" and "today"). The derived decrease in the interest rate of 4 pp, however, conforms well to the empirical estimations (and is even somewhat larger than the standard estimates quoted in the introduction). As far as the wealth-to-income ratios are concerned, the related literature has documented an average increase from around 300%-350% to around 500-550%.[25] The initial wealth-to-income ratio of $\beta = 350\%$ was targeted by my calibration The implied change of the ratio of 250 pp is very much in line with the observed magnitudes (of around 200-250 pp).

Finally, one can look at the share of housing in domestic wealth. The data have documented an increase from around 36% (1970) to 53% (2015).[26] The outcome of the

[25] The most recent data from Bauluz et al. (2022) report an increase in the domestic capital to national income ratio from 1980 to 2018 of 301% to 495% for the US and 301% to 578% for Europe. Referring to a slightly different time interval (1970-2010), Piketty & Zucman (2014), report an increase from 400% to 470% for the US, 305% to 377% for Germany and 359% to 548% for the UK.

[26] These figures correspond to the country-averages in Piketty & Zucman (2014) and Alvaredo et al.

Table 1: A steady-state comparison between the initial and the current situation

Nr.	Case	r	$\boldsymbol{\beta}$	$\frac{\beta_H}{\beta}$	$\frac{P^r_{st}\overline{H}^r_t}{Y_{Nt}}$	$\frac{P^o_{st}\overline{H}^o_t}{Y_{Nt}}$	$\frac{\dot{P}^r_{ht}\overline{H}^r_t}{Y_{Nt}}$	$\frac{\dot{P}^o_{ht}\overline{H}^o_t}{Y_{Nt}}$	$\frac{NDP_t}{Y_{Nt}}$
		Four groups							
1	**Initial**	**9.64%**	**350%**	**46%**	**7%**	**7%**	**2%**	**3%**	**98%**
2	**Today**	**5.69%**	**599%**	**54%**	**5%**	**12%**	**1%**	**4%**	**95%**
3	Lower g	8.52%	370%	45%	7%	7%	1%	2%	95%
4	Lower n	9.25%	363%	47%	7%	7%	2%	3%	98%
5	Higher D	8.86%	376%	48%	7%	7%	2%	3%	97%
6	Lower R	9.57%	353%	46%	7%	7%	2%	3%	98%
7	Lower ρ	9.51%	354%	47%	7%	7%	2%	3%	98%
8	Higher debt	9.93%	375%	39%	7%	7%	2%	2%	99%
9	Higher inequality	7.74%	409%	48%	6%	6%	2%	3%	95%
10	Higher x_m	9.66%	357%	48%	7%	7%	2%	3%	98%
11	Less renters	9.65%	349%	46%	5%	8%	2%	3%	98%
12	Higher κ^{oo}_H, κ^w_H	9.64%	387%	53%	7%	11%	2%	4%	102%

Note: The table shows the implications for various macroeconomic magnitudes when crucial parameters are changed. For the structural parameters I use: $\alpha = 1/3$, $\gamma = 0.17$, $\delta_k = 10\%$, $\delta_h = 2.5\%$, $\eta^r = 1$, $\eta^o = 1.2$, $\xi_d = 5\%$, $\xi_h = 0\%$ and $s_B = 0.0089$. The parameters θ and σ are assumed to adjust such that $g_\varepsilon = g$. For the rest of the parameters I use for the initial state $A = 20$, $R = 65$, $D = 75$, $g = 3\%$, $n = 1\%$, $\rho = 70\%$, $\beta^N_D = 20\%$, $d^w_y = 3$, $\xi_m = 2\%$ and a bequest motive of the top 1% of $s^w_B = 0.093$. For the current situation these values change to $R = 63$, $D = 82$, $g = 1.8\%$, $n = 0.5\%$, $\rho = 60\%$, $\beta^N_D = 70\%$, $d^w_y = 4.5$, $\xi_m = 3\%$ and $s^w_B = 0.207$. The size and change in κ^j_N and κ^j_H are as described in the text. In line 2 all parameters are changed at the same time while in lines 3 to 12 they are changed one at a time.

model—a shift from 46% to 54%—is again broadly in line with the level and direction of the observed data (even though the implied increase is somewhat lower than the empirical trend). I will discuss below why the assumptions about the outright owners are crucial in order to capture this dimension of the data.

In lines 3 to 12 of Table 1 I separate the contributions of the various factors. As one can see, the increase in inequality (captured by the changes in s_B^w and in d_y^w) has the largest effect on the equilibrium values followed by the decrease in the growth rate g, the increase in life expectancy D and the decrease in n. The total effect is thereby due to shifts of both the wealth supply and the wealth demand schedule. It is instructive to look at a graphical illustration of this effect. Panel (a) of Figure 2 shows that the increase in life expectancy mainly affects the wealth supply schedule and has only a tiny impact on wealth demand (via the value of the housing stock that depends on aggregate demand). This upward shift of the wealth supply curve increases equilibrium wealth while decreasing the equilibrium interest rate. A similar picture emerges for a decrease in g although now the effect on the wealth demand schedule is somewhat larger (although it still only works via housing wealth). In general, one can say that the empirically observed pattern of decreasing interest rates and increasing wealth ratios requires that the upward shift in the wealth supply schedule dominates a potential downward movement in wealth demand.

Table 1 also reports the results for the various components of national income. In particular, the share of actual and of imputed rents (both as a ratio of non-housing output Y_{Nt}) come out as around 7% each ($\frac{P^r_{st}\overline{H}^r_t}{Y_{Nt}} = 6.6\%$, $\frac{P^o_{st}\overline{H}^o_t}{Y_{Nt}} = 7.1\%$) and the share of capital gains on the rented and the owned stock as $\frac{\dot{P}^r_{ht}\overline{H}^r_t}{Y_{Nt}} = 2\%$ and $\frac{\dot{P}^o_{ht}\overline{H}^o_t}{Y_{Nt}} = 2.7\%$, respectively. The total depreciation of physical capital and the housing stock, on the other hand, amount to (not shown): $\delta_k\beta_K^N = 16.4\%$, $\delta_h\beta_H^N = 3.9\%$.[27] Therefore the ratio of GDP_t to Y_{Nt} is given by: $1 + 0.137 + 0.047 = 1.184$ while the ratio of NDP_t to Y_{Nt} is: $1.184 - 0.164 - 0.039 = 0.98$ which implies that $NDP_t/GDP_t = 0.828$.

(2018). There exist, however, considerable differences between countries both for the level and for the changes where the share moved from 36% to 40% for the US, from 27% to 55% for the UK and from 42% to 64% for Germany.

[27] Expressed as a share of GDP this amounts to 13.9% and 3.4%, respectively. Empirical data show that capital depreciation amounts to values between 11% (France, US) and 19% (Japan) of GDP with an average of around 14%. This is in line with the calibration of the model.

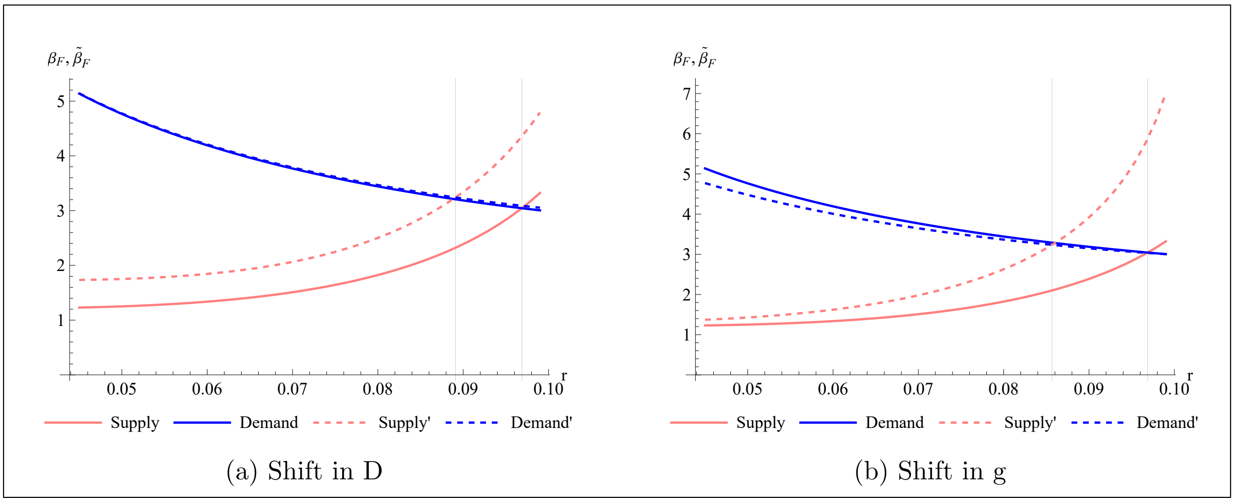


Figure 2: The figures show the effect of changes in life expectancy from $D = 75$ to $D = 82$ (panel (a)) and in the growth rate from $g = 3\%$ to 1.8% (panel (b)) on the wealth supply schedule $\widetilde{\beta}_F$ and the wealth demand schedule $\beta_F = \beta_K + \beta_{Hr} + \beta_M + \beta_D$. The rest of the parameters are like in Table 1.

5.3 Additional variables

The model has also implications for a number of additional magnitudes that can be compared to their real-world counterparts in order to assess the plausibility of the benchmark model and the differences to the competing models. These additional variables are the interest rate spread, the capital-to-income ratio, the mortgage volume, the aggregate savings rates, the capital gain savings and the inheritance flows. Table 2 reports the implications of the different models for these empirical regularities. Beside Models 1 to 3, the table now also reports as a comparison the results for two cases without housing. Model 4 uses the same strength of the bequest model as the two-group model with housing (Model 3) while Model 5 uses a calibration for s_B and s_B^w that targets an initial wealth-to-income ratio of 350% and a top 1% wealth share that increases from 28% to 35%.

- Interest rates spreads: As a first observation one can note that the benchmark model implies interest rate differentials that are well aligned with the empirically observed patterns. Although this is mostly due to the exogenously assumed risk premia it is nevertheless noteworthy since it gives the model a realistic foundation. In particular, the current equilibrium in line 2 of Table 2 has $r_k = r_h = 7.2\%$, $r_m = 4.2\%$, $r_d = 2.2\%$ which looks reasonable. Furthermore, the *average* rate of return on housing is a weighted average of the rate of return on residential investments r_h and on owner-occupied housing r_m. It decreases from the initial to the current period

Table 2: Savings rates and inheritance flows in various models

Nr.	Case	r	β	$\frac{\beta_H}{\beta}$	$\bar{s}$	$\bar{s}^{net}$	$\bar{s}^{NA}$	$\bar{s}^{NA,net}$	b_{Fy}	b_y
		Model 1: Four groups								
1	**Initial**	**9.64%**	**350%**	**46%**	**29.9%**	**15.3%**	**28.2%**	**12.9%**	**5.9%**	**6.6%**
2	**Today**	**5.69%**	**599%**	**54%**	**33.9%**	**15.1%**	**31.4%**	**11.%**	**7.1%**	**9.2%**
		Model 2: Three groups (no outright owners, $\kappa_N^{oo} = 0$)								
3	**Initial**	9.43%	350%	46%	28.2%	13.3%	26.6%	10.9%	6.9%	6.9%
4	**Today**	6.52%	480%	44%	27.4%	9.4%	25.8%	7.%	8.3%	8.3%
		Model 3: Two groups (no owners, $\kappa_N^{oo} = \kappa_N^{om} = 0$)								
5	**Initial**	9.27%	350%	44%	28.5%	13.3%	28.5%	13.2%	6.8%	6.8%
6	**Today**	6.19%	476%	40%	28.%	9.3%	27.9%	9.2%	7.9%	7.9%
		Model 4: Two groups (no housing, $\gamma = 0$; $s_B = 0.0128$ and $s_B^w = 0.1049$)								
7	**Initial**	7.68%	249%	0%	25.7%	9.%	25.7%	9.%	3.9%	3.9%
8	**Today**	4.04%	369%	0%	26.9%	6.4%	26.9%	6.4%	3.1%	3.1%
		Model 5: Two groups (no housing, $\gamma = 0$; $s_B = 0.0454$ and $s_B^w = 0.597$)								
9	**Initial**	3.26%	350%	0%	34.2%	12.9%	34.2%	12.9.%	4.2%	4.2%
10	**Today**	-1.32%	647%	0%	43.1%	12.4%	43.1%	12.4%	5.5%	5.5%

Note: See Table 3. The last four lines now contain the results for two models without housing. The first is based on the same bequest motives as Model 3 ($s_B = 0.0128$ and $s_B^w = 0.1049$ ($s_B^w = 0.153$)) while the second model use a calibration that targets an initial wealth-to-income ratio of 350% and a top 1% wealth share that increases from 28% to 35%. This implies $s_B = 0.0454$ and $s_B^w = 0.597$ ($s_B^w = 1.45$).

by 4.3 pp (from 9.1% to 4.8%) while the spread to r_k increase from 1.1 pp to 2.4 pp which is larger than the assumed increase in ξ_m. This effect is due to the increase in the share of mortgage lending which decreases the average return on housing. This is in line with the results in Jordà et al. (2019) who report the return on equity and housing as 8.2% and 6.4%, respectively, for the post-1950 period and 9.1% and 5.5%, respectively, for the post-1980 period. Although these data involve different periods then the ones underlying Table 3 they document a comparable trend in the spread between investments in equity and in housing.

The model without housing has difficulties to match the levels and patterns of interest rates. If one calibrates the bequest motives in a way as to target $\beta = 350\%$ (see line 9) then this implies a low interest rate of $r = 3.3\%$ with an only slightly higher return of equity of $r_k = 3.6\%$ and a government bond rate of $r_d = -2.6\%$. This is further reduced to negative territory for the current situation with $r = -1.3\%$, $r_k = -0.5\%$ and $r_d = -5.5\%$. These values seem implausibly low and are not in line with the empirical data.

- Mortgage volume: Related to the first point, the benchmark model involves an increase in the share of total mortgages to GDP from 36% to 74%, i.e. an increase by 38 pp (not shown). This is in line with the data from Jordà et al. (2016) who have reported that mortgages increased sharply over the last century—from about 20% of GDP at the beginning of the 20th century to about 70% today with a particularly large increase since the 1970ies. If one would abstract from outright owners (see Model 2) then this three group model is associated with an increase in the share of mortgages from 73% to 114% which does not conform to the empirical data.

- Capital-to-income ratio: Due to the existence of housing there is a disconnect between the development of the wealth-to-income and the capital-to-income ratios in all models that include housing. While the former increases considerably (by 250 pp) the latter undergoes only a slight movement from 168% to 204% (not shown). This is in line with the data in Piketty & Zucman (2014), Bonnet et al. (2014), Bonnet et al. (2021) that also report an average capital-income ratio around 200% which barely moves over time.

  For the models without housing, on the other hand, wealth and capital are almost identical (the difference being the stock of government debt). For Model 5, e.g., the capital-to-income ratio increases by more than 200 pp (from 324% to 539%).

- Aggregate savings rate: Table 2 reports the development of aggregate savings rates between the two points in time. At first sight, the increase in the gross savings rate from 29.9% to 33.9% seems larger than could be observed in the data. For example, in a similar calibration exercise based on the Solow model, Mankiw (2022) uses an increase in the gross savings rate from 22% to 25%.[28] One has to take into account, however, that these official savings rates exclude the capital gains. If one subtracts $\dot{P}^{r}_{ht}\overline{H}^{r}_{t}$ and $\dot{P}^{r}_{ht}\overline{H}^{o}_{t}$ in both the numerator and the denominator of the savings rate one gets a "national-account gross savings rate" (see (37)) of $\overline{s}^{NA} = 28.2\%$ in the initial and $\overline{s}^{NA} = 31.4\%$ in the current situation. This is still larger than the observed magnitudes but at least closer to the range of empirically observed data. For the *net* savings rate according to the traditional concept the results even indicate a *decrease*. Model 5 without housing involves an unreasonable level and increase in the gross savings rates (from 34.2% to 43.1%).

- Inheritance flow: Finally, one can also look at the inheritance flows (the ratio of annual bequests to national income). The flow of financial bequests comes out as $b_{Fy} = 5.9\%$ in the initial situation. There do not seem to exist reliable data on the inheritance flows among the group of advanced countries. Alvaredo et al. (2017) report a value of around 7% for France, UK and Germany in the 1980ies which is thus slightly above the implied value of the model. For the four group model it is, however, important to note that there are now two kinds of inheritances: the financial bequests B_{Ft} and the bequest of the housing stock of the outright owners B_{Ht}. As shown in appendix A.6.3 the latter element can be simply calculated as the product of the value of this part of the housing wealth and the mortality rate. As can be seen in the last column of Table 2 this increases the implied inheritance flow from 5.9% to 6.6% which is closer to the observed magnitudes. In the steady-state comparison this total flow is predicted to increase from 6.6% to 9.2%. This increase is somewhat below the observed data for the above-mentioned subsample of countries where the shift is reported from around 7% to around 11%. In comparison to the results of the other models, the implications of Model 1 are, however, most closely aligned with the empirical pattern.

- Capital gain saving: Finally, one can point to a stylized fact that has been recently uncovered based on administrative data from the Norwegian wealth register. In

[28]In particular, the rate used by Mankiw (2022) is based on Worldbank data and refers to the global gross savings rates during the first half and the second half of the period from 1975-2022, respectively.

particular, Fagereng et al. (2019) show that the saving rates net of capital gains are approximately flat across the wealth distribution while the saving rates including capital gains increase systematically with wealth. The authors explain this stylized fact by noting that "wealthier households own assets that experience capital gains which they hold on to instead of selling them off to consume". This stylized fact is also present in the framework of my model once it is assumed that the size of the houses inherited by outright owners is uncorrelated with the level of permanent income y_{Lti}.

5.4 The role of owners and renters

A crucial ingredient of the benchmark model and the reason why it aligns quite well with a number of empirical facts is the assumption that the group of owners consists of two subgroups and that the housing stock under the control of outright owners has increased over time. This can be seen by looking at Table 3. Model 1 corresponds to the benchmark case of Table 1 while Model 2 refers to the case where one abstracts from outright owners and assumes instead that all owners are financing their home purchases with mortgages. The level and the decrease in the interest rate is similar to the benchmark case but the model now has problems to match the empirical data along two dimensions. First, the resulting increase in the wealth-to-income ratio is considerably smaller than before (130 pp vs. 250 pp) and second, the model now implies a counterfactual *decrease* in the share of housing wealth (from 46% to 44%) instead of an increase as in the benchmark model. A very similar conclusion emerges if one neglects owners altogether and focuses on a model with only two groups (renters and the top 1%) as shown in Model 3 of Table 3.

In order to understand the reason behind this crucial result it is best to start with the case that abstracts from outright owners. Using equations (14) to (16) the housing-to-capital ratio can be written as:

$$\frac{\beta_H}{\beta_K} = \frac{\frac{1}{Y_{Nt}}\left(\frac{P^r_{st}\overline{H}^r_t}{r_h+\delta_h-\widetilde{g}} + \frac{P^o_{st}\overline{H}^{om}_t}{r_m+\delta_h-\widetilde{g}}\right)}{\frac{\alpha}{r_k+\delta_k}}.$$

The first thing to note is that the share of expenditures on rental ($\frac{P^r_{st}\overline{H}^r_t}{Y_{Nt}}$) and owned houses ($\frac{P^o_{st}\overline{H}^{om}_t}{Y_{Nt}}$) typically does not react much to changes in the economic structure. In particular, if there are shifts in the housing stock $\overline{H}^r_t$ ($\overline{H}^{om}_t$) then P^r_{st} (P^{om}_{st}) will simply adjust such as to leave the total expenditure shares basically unchanged. But even for

Table 3: The results for various assumptions about the housing market

Nr.	Case	r	β	$\frac{\beta_H}{\beta}$	$\frac{P^r_{st}\overline{H}^r_t}{Y_{Nt}}$	$\frac{P^o_{st}\overline{H}^o_t}{Y_{Nt}}$	$\frac{\dot{P}^r_{ht}\overline{H}^r_t}{Y_{Nt}}$	$\frac{\dot{P}^o_{ht}\overline{H}^o_t}{Y_{Nt}}$	$\frac{NDP_t}{Y_{Nt}}$
		Model 1: Four groups							
1	**Initial**	**9.64%**	**350%**	**46%**	**7%**	**7%**	**2%**	**3%**	**98%**
2	**Today**	**5.69%**	**599%**	**54%**	**5%**	**12%**	**1%**	**4%**	**95%**
		Model 2: Three groups (no outright owners, $\kappa_N^{oo} = 0$)							
3	**Initial**	9.43%	350%	46%	7%	7%	2%	3%	98%
4	**Today**	6.52%	480%	44%	5%	8%	1%	2%	94%
		Model 3: Two groups (no owners, $\kappa_N^{oo} = \kappa_N^{om} = 0$)							
5	**Initial**	9.27%	350%	44%	13%	0%	4%	0%	97%
6	**Today**	6.19%	476%	40%	13%	0%	3%	0%	92%

Note: The table shows the results for different assumptions about the housing market. Lines 1 and 2 contain the benchmark results from Table 1. The parameter values are the same as there. In Model 2 there are no outright owners ($\kappa_N^{oo} = 0$) except for the top 1%. The bequest motive is adjusted such as to target $\beta = 350\%$ and a wealth share of 28% (35%) for the top 1% in the initial (current) situation. This implies $s_B = 0.0122$ and $s_B^w = 0.1$ ($s_B^w = 0.138$). For Model 3 there are only renters (except among the top 1%) and the calibration uses $s_B = 0.0128$ and $s_B^w = 0.1049$ ($s_B^w = 0.153$).

constant expenditure shares there can be huge changes in the house prices P_{ht}^r and P_{ht}^o if the interest rates r_h and r_m change. The effect on the *ratio* $\frac{\beta_H}{\beta_K}$, however, will depend on whether this impact on the house prices is weaker or stronger than the impact on $\frac{\alpha}{r_k+\delta_k}$. In appendix B I use a stylized analytical example to show that under certain conditions this boils down to the question whether the main force behind the decline in the interest is a decrease in g or a decrease in n with $\frac{\beta_H}{\beta_K} = 0$ for $\frac{\dot{g}}{g} = \frac{\dot{n}}{n}$. This conclusion, however, is different in the presence of outright owners. The outright owners do not adjust their holdings of housing and for them an increase in P_{ht}^o has a first-order effect on their housing wealth. This mechanism is reflected in the large increase in the value of $\frac{P^o_{st}\overline{H}^o_t}{Y_{Nt}}$ (from 7% to 12%), of $\frac{\dot{P}^o_{ht}\overline{H}^o_t}{Y_{Nt}}$ (from 3% to 4%) and in particular in the share of housing wealth in total wealth. The crucial role of outright owners is also reflected in the results of the lower part of Table 1 where it appears that other changes in the housing arrangements (the decrease in the share of renters and the change in the risk discount ξ_m) have by themselves only a minor effect on the equilibrium interest rate and the equilibrium wealth-to-income ratio.

The expression for $\frac{\beta_H}{\beta_K}$ also suggests a number of other parameter changes that could be invoked in order to cause a change in the share of housing wealth. These include adjustments in the non-housing sector (e.g., a smaller capital share α, a higher depreciation rate δ_k), the housing sector (e.g., a larger preference for housing γ, a slower increase in housing supply χ, a lower depreciation rate δ_h, higher risk discounts ξ_m and ξ_h), and the preference side (e.g., a higher degree of intertemporal substitution θ, a larger bequest motive s_B). In appendix D, I discuss a number of alternative specifications along these lines and I explore whether they could be used to match the empirical data (in particular the increasing share of housing wealth) without assuming a change in the role of outright owners. I conclude that while some of these specifications lead to an increase in the share of housing wealth, they typically exhibit deficits along other dimensions or require assumptions incongruent with empirical observations. The only exception is a decrease in δ_h which also aligns quite well with most empirical regularities and which could be interpreted as an increase in the attractiveness of homeownership (possibly due to changes in the regulatory or tax environment).

5.5 Comparison to the related literature

It is also interesting to compare the results of this paper to the ones of the related literature, in particular to the papers by Summers & Rachel (2019) and Platzer & Peruffo (2022). The former is based on a perpetual youth model that distinguishes—similar to Gertler (1999)—between workers and retirees. The latter, on the other hand, also uses a OLG model with workers and pensioners but in addition also assumes that agents differ in their income processes and possibly also in their savings preferences. Both models, however, also use steady-state comparisons to study the sources of the decline in interest rates and they use a similar set of driving variables. In particular, Platzer & Peruffo (2022) compare the steady state of the situation around 1975 with the situation around 2015 while Summers & Rachel (2019) look at the time span from 1970 to 2017. Both papers include productivity growth, population growth, life expectancy and an increase in inequality as potential factors behind the decline in the interest rate and they also include the development of public debt.

Table 4 documents that despite the differences in the theoretical set-up and in the empirical calibration of the three models[29] they agree in the ranking (though not in the precise numerical values) of the most important factors behind the decline in interest

[29] In particular, the models use different values to describe the initial and current state. Summers & Rachel (2019) use, e.g., a change from $g = 1.51\%$ and $n = 1.35\%$ to $g = 0.7\%$ and $n = 0.5\%$ while Platzer & Peruffo (2022) assume a drop from $g = 1.5\%$ and $n = 1.92\%$ to $g = 0.7\%$ and $n = 1.15\%$. Note, however, that the change in the total growth rate $(g + n)$ is similar in both papers and given by 1.71% and 1.57%, respectively, which is also close to my assumptions where the drop amounts to $(4\% - 2.3\%) = 1.7\%$.

rates. The the rise in inequality, the slowdown in productivity growth and the increase in life expectancy are seen as the main drivers followed by the decline in population growth.

Table 4: Decomposition of the Decline in the Neutral Real Interest Rate

Variable	**RS '19**	**PP '22**	**This paper** $\gamma = 0$ (2 Groups)	**This paper** $\gamma = 0.17$ (2 Groups)	**This paper** $\gamma = 0.17$ (4 Groups)
TFP growth (g)	-1.8	-1.00	-1.01	-1.13	-1.12
Pop. growth (n)	-0.6	-0.25	-0.43	-0.38	-0.39
Longer retirement (D)	-1.1	-0.46	-1.1	-0.75	-0.78
Length of working life (R)	-0.1	–	-0.18	-0.07	-0.07
Replacement rate (ρ)	–	–	-0.33	-0.12	-0.13
Inequality (s_B^w and d_y^w)	-0.7	-0.70	-0.62	-0.71	-1.9
Public Debt (β_D^N)	+3.6	+0.31	+0.77	+0.36	+0.29
Interactions	-1.1	-0.06	-0.13	0.02	0.12
Other factors	–	0.00	–	–	0.03
Total	-1.8	-2.16	-3.03	-2.78	-3.95

Note: The table summarizes the results in Summers & Rachel (2019) (RS'19, Table 7), Platzer & Peruffo (2022) (PP'22, Table 5) and three version of the present paper (two groups with and without housing and the four-group model of the last section). All values are in percentage points. The category "public debt" for Summers & Rachel (2019) also includes the increase in implicit government debt due to changes in Social Security and old-age health care. Increases in explicit public debt alone amount to only +1.2 pp. In Platzer & Peruffo (2022) the category "other factors" includes out-of-pocket medical expenses, changes in the labor share and exogenous government spending. Their individual contributions are evaluated as -0.14 pp, $+0.11$ pp and $+0.03$ pp thus summing up to zero. In the last column the category "other factors" refers to the changes in ξ_m and in the share of renters and outright owners.

6 Conclusion

In this paper I presented a model of wealth accumulation that includes a housing sector. Households' preference for housing stock give rise to a positively valued housing stock that can be used as an additional investment vehicle besides the usual stock of physical capital. The benchmark model is based on a four-group. The bottom 99% are assumed to differ from the top 1% in their level of lifetime income and in the strength of their bequest motive. Furthermore it is assumed that the bottom 99% contain three groups of dwellers: renters, owners with mortgages and outright owners who are assumed to reside in houses that are passed on from generation to generation. When comparing the steady state of the model calibrated to parameter values representing the situation around

1980 with the steady state around 2018 the model is quite successful in reproducing the important stylized facts. In particular, the model implies a decline in the equilibrium interest rate of 4 pp, an increase in the wealth-to-income ratio of 250 pp and an increase in the share of housing wealth by 8 pp. All of these results are close to the empirically observed values. What is more, the extended model is also able to account for a wedge between safe and risky interest rates, for an increase in the inheritance flows and for a proliferation of mortgages. The paper ended with a discussion of why the assumption about the behavior of outright owners is crucial for capturing these developments and why models with a different housing structure (or no housing) fall short along a number of dimensions.

References

Alvaredo, F., Chancel, L., Piketty, T., Saez, E. & Zucman, G. (2018), *World Inequality Report 2018*, Belknap Press.

Alvaredo, F., Garbinti, B. & Piketty, T. (2017), 'On the Share of Inheritance in Aggregate Wealth: Europe and the USA, 1900-2010', *Economica* **84**(334), 239–260.

Auclert, A., Malmberg, H., Martenet, F. & Rognlie, M. (2021), Demographics, Wealth, and Global Imbalances in the Twenty-First Century, Technical report, National Bureau of Economic Research.

Bauluz, L., Novokmet, F. & Schularick, M. (2022), The Anatomy of the Global Saving Glut, Technical report.

Bernanke, B. S. (2005), The Global Saving Glut and the U.S. Current Account Deficit, Technical report.

Bonnet, O., Bono, P.-H., Chapelle, G. & Wasmer, E. (2014), Does Housing Capital Contribute to Inequality? A Comment on Thomas Piketty's Capital in the 21st Century, Discussion Paper 2014-07, Sciences Po Departement of Economics.

Bonnet, O., Chapelle, G., Trannoy, A. & Wasmer, E. (2021), 'Land Is Back, It Should Be Taxed, It Can Be Taxed', *European Economic Review* **134**, 103696.

Borri, N. & Reichlin, P. (2018), 'The Housing Cost Disease', *Journal of Economic Dynamics and Control* **87**(C), 106–123.

Caballero, R. J., Farhi, E. & Gourinchas, P.-O. (2017), 'The Safe Assets Shortage Conundrum', *Journal of Economic Perspectives* **31**(3), 29–46.

Chambers, D., Spaenjers, C. & Steiner, E. (2021), 'The Rate of Return on Real Estate: Long-Run Micro-Level Evidence', *The Review of Financial Studies* **34**(8), 3572–3607.

Chen, P., Karabarbounis, L. & Neiman, B. (2017), 'The Global Rise of Corporate Saving', *Journal of Monetary Economics* **89**(C), 1–19.

Eggertsson, G. B., Mehrotra, N. R. & Robbins, J. A. (2019), 'A Model of Secular Stagnation: Theory and Quantitative Evaluation', *American Economic Journal: Macroeconomics* **11**(1), 1–48.

Eichholtz, P., Korevaar, M., Lindenthal, T. & Tallec, R. (2021), 'The Total Return and Risk to Residential Real Estate', *The Review of Financial Studies* **34**(8), 3608–3646.

Fagereng, A., Holm, M. B., Moll, B. & Natvik, G. (2019), Saving Behavior Across the Wealth Distribution: The Importance of Capital Gains, Working Paper 26588, National Bureau of Economic Research.

Favilukis, J., Ludvigson, S. C. & van Nieuwerburgh, S. (2017), 'The Macroeconomic Effects of Housing Wealth, Housing Finance, and Limited Risk Sharing in General Equilibrium', *Journal of Political Economy* **125**(1), 140–223.

Fessler, P. & Schürz, M. (2021), Structuring the Analysis of Wealth Inequality Using the Functions of Wealth: A Class-Based Approach, *in* 'Measuring Distribution and Mobility of Income and Wealth', NBER Chapters, National Bureau of Economic Research, Inc, pp. 221–248.

Garbinti, B., Goupille-Lebret, J. & Piketty, T. (2020), 'Accounting for Wealth-Inequality Dynamics: Methods, Estimates, and Simulations for France', *Journal of the European Economic Association* **19**(1), 620–663.

Gertler, M. (1999), 'Government Debt and Social Security in a Life-cycle Economy', *Carnegie-Rochester Conference Series on Public Policy* **50**(1), 61–110.

Gordon, R. J. (2014), The Turtle's Progress: Secular Stagnation Meets the Headwinds, *in* R. Baldwin & C. Teulings, eds, 'Secular Stagnation', CEPR, United Kingdom, pp. 47–60.

Grossmann, V., Larin, B., Löffad, H. T. & Steger, T. (2021), 'Distributional Consequences of Surging Housing Rents', *Journal of Economic Theory* **196**, 105275.

Grossmann, V., Larin, B. & Steger, T. M. (2021), Das House-Kapital, FSES Working Papers 523, Faculty of Economics and Social Sciences, University of Freiburg/Fribourg Switzerland.

Iacoviello, M. & Pavan, M. (2013), 'Housing and Debt Over the Life Cycle and Over the Business Cycle', *Journal of Monetary Economics* **60**(2), 221–238.

Jordà, Ò., Knoll, K., Kuvshinov, D., Schularick, M. & Taylor, A. M. (2019), 'The Rate of Return on Everything, 1870–2015', *The Quarterly Journal of Economics* **134**(3), 1225–1298.

Jordà, Ò., Schularick, M. & Taylor, A. M. (2016), 'The Great Mortgaging: Housing Finance, Crises and Business Cycles', *Economic Policy* **31**(85), 107–152.

Justiniano, A., Primiceri, G. E. & Tambalotti, A. (2019), 'Credit Supply and the Housing Boom', *Journal of Political Economy* **127**(3), 1317–1350.

Kaplan, G., Mitman, K. & Violante, G. L. (2020), 'The Housing Boom and Bust: Model Meets Evidence', *Journal of Political Economy* **128**(9), 3285–3345.

Knoll, K., Schularick, M. & Steger, T. (2017), 'No Price Like Home: Global House Prices, 1870-2012', *American Economic Review* **107**(2), 331–53.

Kuhn, M., Schularick, M. & Steins, U. I. (2020), 'Income and Wealth Inequality in America, 1949–2016', *Journal of Political Economy* **128**(9), 3469–3519.

Laubach, T. & Williams, J. C. (2003), 'Measuring the Natural Rate of Interest', *The Review of Economics and Statistics* **85**(4), 1063–1070.

Mankiw, N. G. (2022), 'Government Debt and Capital Accumulation in an Era of Low Interest Rates', *Brookings Papers on Economic Activity* **2022**, 219–231.

McKay, A. & Wieland, J. F. (2021), 'Lumpy Durable Consumption Demand and the Limited Ammunition of Monetary Policy', *Econometrica* **89**(6), 2717–2749.

Mian, A. R., Straub, L. & Sufi, A. (2021*a*), 'What explains the decline in r*? Rising income inequality versus demographic shifts', (2021-12).

Mian, A., Straub, L. & Sufi, A. (2021*b*), 'Indebted Demand', *The Quarterly Journal of Economics* **136**(4), 2243–2307.

Modigliani, F. (1986), 'Life Cycle, Individual Thrift, and the Wealth of Nations', *American Economic Review* **76**(3), 297–313.

Piazzesi, M. & Schneider, M. (2016), Housing and Macroeconomics, Vol. 2 of *Handbook of Macroeconomics*, Elsevier, pp. 1547–1640.

Piketty, T. (2010), On the Long-Run Evolution of Inheritance: France 1820-2050, PSE Working Papers 2010-12, Paris School of Economics.

Piketty, T. (2011), 'On the Long-Run Evolution of Inheritance: France 1820–2050', *The Quarterly Journal of Economics* **126**(3), 1071–1131.

Piketty, T. & Zucman, G. (2014), ‘Capital is Back: Wealth-Income Ratios in Rich Countries 1700-2010’, *The Quarterly Journal of Economics* **129**(3), 1255–1310.

Platzer, J. & Peruffo, M. (2022), ‘Secular Drivers of the Natural Rate of Interest in the United States: A Quantitative Evaluation’, *IMF Working Papers* **2022**(030).

Poterba, J. M. (1984), ‘Tax Subsidies to Owner-Occupied Housing: An Asset-Market Approach’, *The Quarterly Journal of Economics* **99**(4), 729–752.

Rachel, L. & Smith, T. (2015), Secular Drivers of the Global Real Interest Rate, Bank of England Working Papers 571, Bank of England.

Robbins, J. (2018), Capital Gains and the Distribution of Income in the United States, mimeo, Brown University.

Rognlie, M. (2016), ‘Deciphering the Fall and Rise in the Net Capital Share: Accumulation or Scarcity?’, *Brookings Papers on Economic Activity* **2015**(1), 1–69.

Sommer, K. & Sullivan, P. (2018), ‘Implications of US Tax Policy for House Prices, Rents, and Homeownership’, *American Economic Review* **108**(2), 241–274.

Summers, L. H. & Rachel, L. (2019), On Falling Neutral Real Rates, Fiscal Policy and the Risk of Secular Stagnation, *in* ‘Brookings Papers on Economic Activity BPEA Conference Drafts, March’, Vol. 7, pp. 1–54.

Svensson, L. E. (2023), Are Swedish House Prices Too High? Why the Price-to-Income Ratio Is a Misleading Indicator, mimeo.

Appendices

A Details of the model

A.1 National accounting

As noted in section 2.3 total wealth and the various subaggregates of wealth demand and wealth supply can be related to each of the income concepts Y_{Nt}, GDP_t and NDP_t (and also to GNP_t and NNP_t if one would include an open economy structure). The use of one or the other income concept depends on its usefulness and/or tractability. In the text I have shown the formulas for β_t^N and β_t^{NDP}, i.e. using the domestic production (excluding housing services) Y_{Nt} and the net domestic product NDP_t, respectively. The transformation between the various concepts is straightforward using appropriate multiplicative factors. These factors can also be applied to all different "partitions" of wealth. For example:

$$\beta_{Kt}^N = \frac{K_t}{Y_{Nt}}, \beta_{Ht}^N = \frac{P_{ht}^r \overline{H}_t^r + P_{ht}^o \overline{H}_t^o}{Y_{Nt}}, \beta_{Dt}^N = \frac{D_t}{Y_{Nt}}, \beta_{Mt}^N = \frac{M_t}{Y_{Nt}}. \tag{44}$$

If one wants to calculate the wealth-to-GDP-ratio then one has to take the definition (10) into account $GDP_t = Y_{Nt} + P_{st}^r \overline{H}_t^r + P_{st}^o \overline{H}_t^o + \dot{P}_{ht}^r \overline{H}_t^r + \dot{P}_{ht}^o \overline{H}_t^o$. It thus holds for any wealth concept $x \in \{K, H, H_r, H_o, D, M\}$ that:

$$\beta_{xt}^{GDP} = \beta_{xt}^N \frac{Y_{Nt}}{GDP_t} = \beta_{xt}^N \frac{1}{1 + \frac{P_{st}^r \overline{H}_t^r}{Y_{Nt}} + \frac{P_{st}^o \overline{H}_t^o}{Y_{Nt}} + \frac{\dot{P}_{ht}^r \overline{H}_t^r}{Y_{Nt}} + \frac{\dot{P}_{ht}^o \overline{H}_t^o}{Y_{Nt}}}. \tag{45}$$

Focusing on the net domestic product the parallel relation holds:

$$\beta_{xt}^{NDP} \equiv \beta_{xt} = \beta_{xt}^N \frac{Y_{Nt}}{NDP_t} \tag{46}$$

which has already been shown for $\beta_t^{NDP} = \beta_t$ in section 2.3.

In equation (18) of the paper I state that in the steady state with $r_{kt} = r_{ht} = r_{mt} = r$ and $\beta_D^N = 0$ the ratio of net domestic product to non-housing output can also be written as $\frac{NDP_t}{Y_{Nt}} = 1 + r\beta^N - \alpha = 1 + \alpha\frac{\beta_H}{\beta_K} - \delta_k\beta^N$. This can be derived as follows:

$$\begin{aligned}
\frac{NDP_t}{Y_{Nt}} &= 1 + \frac{P^r_{st}\overline{H}^r_t}{Y_{Nt}} + \frac{P^o_{st}\overline{H}^o_t}{Y_{Nt}} + \frac{\dot{P}^r_{ht}\overline{H}^r_t}{Y_{Nt}} + \frac{\dot{P}^o_{ht}\overline{H}^o_t}{Y_{Nt}} - \frac{\delta_h P^r_{ht}\overline{H}^r_t}{Y_{Nt}} - \frac{\delta_h P^o_{ht}\overline{H}^o_t}{Y_{Nt}} - \frac{\delta_k K_t}{Y_{Nt}} \\
&= 1 + r_{ht}\frac{P^r_{ht}\overline{H}^r_t}{Y_{Nt}} + r_{mt}\frac{P^o_{ht}\overline{H}^o_t}{Y_{Nt}} - \frac{\delta_k K_t}{Y_{Nt}} \qquad (47)
\end{aligned}$$

where the transformation follows from equations (4) $\left(P^r_{st} = P^r_{ht}\left(r_{ht} + \delta_h - \frac{\dot{P}^r_{ht}}{P^r_{ht}}\right)\right)$ and (5) $\left(P^o_{st} = P^o_{ht}\left(r_{mt} + \delta_h - \frac{\dot{P}^o_{ht}}{P^o_{ht}}\right)\right)$ which imply that $\overline{H}^r_t\left(P^r_{st} + \dot{P}^r_{ht} - \delta_h P^r_{ht}\right) = r_{ht}P^r_{ht}\overline{H}^r_t$ and $\overline{H}^o_t\left(P^o_{st} + \dot{P}^o_{ht} - \delta_h P^o_{ht}\right) = r_{mt}P^o_{ht}\overline{H}^o_t$. In the next step one can use $\beta^N = \beta^N_H + \beta^N_K$, where $\beta^N_H = \frac{P^r_{ht}\overline{H}^r_t + P^o_{ht}\overline{H}^o_t}{Y_{Nt}}$ and $\beta^N_K = \frac{K_t}{Y_{Nt}}$. One can thus write:

$$\begin{aligned}
\frac{NDP_t}{Y_{Nt}} &= 1 + r\frac{P^r_{ht}\overline{H}^r_t}{Y_{Nt}} + r\frac{P^o_{ht}\overline{H}^o_t}{Y_{Nt}} - \frac{\delta_k K_t}{Y_{Nt}} \\
&= 1 + r\left(\beta^N - \beta^N_K\right) - \delta_k\beta^N_K = 1 + r\beta^N - (r + \delta_k)\beta^N_K \\
&= 1 + r\beta^N - \alpha,
\end{aligned}$$

where I use the steady state condition $r_{kt} = r_{ht} = r_{mt} = r$ together with $r_{kt} = \alpha\frac{Y_{Nt}}{K_t} - \delta_k = \alpha\frac{1}{\beta^N_K} - \delta_k$ from equation (2) which implies that $(r + \delta_k)\beta^N_K = \alpha$. Using $r = \frac{\alpha}{\beta^N_K} - \delta_k$ the expression above can also be written as:

$$\begin{aligned}
\frac{NDP_t}{Y_{Nt}} &= 1 + r\beta^N_H - \delta_k\beta^N_K = 1 + \beta^N_H\left(\frac{\alpha}{\beta^N_K} - \delta_k\right) - \delta_k\beta^N_K \\
&= 1 + \alpha\frac{\beta^N_H}{\beta^N_K} - \delta_k\left(\beta^N_K + \beta^N_H\right) = 1 + \alpha\frac{\beta_H}{\beta_K} - \delta_k\beta^N,
\end{aligned}$$

where in the last line I use $\frac{\beta^N_H}{\beta^N_K} = \frac{\beta_H}{\beta_K}$ since the income concept used in the denominator drops out (and where $\beta_K = \frac{K_t}{NDP_t}$, $\beta_H = \frac{P^r_{ht}\overline{H}^r_t + P^o_{ht}\overline{H}^o_t}{NDP_t}$).

A.2 Pension system

In section 3.3 of the paper I specify the equations for the PAYG pension system under the assumption of no seniority wages (i.e. $y_{Lti}(a) = y_{Lti}$ for $a \in [A, R[$). In this case one can write the flows of the pension system in time period t in the form of equation (22). Here I want to show the expression that is valid for a general wage profile where one has to use the cohort-specific formulation. In particular:

$$y^x_{Li}(a) = (1-\tau_\rho)\mathrm{y}^x_{Li}(a) \text{ for } a \in [A,R[$$
$$y^x_{Li}(a) = \rho(1-\tau_\rho)\iota_i \; \overline{\mathrm{y}}_{L,x+a} \text{ for } a \in [R,D], \tag{48}$$

where $\mathrm{y}^x_{Li}(a)$ stands for the pre-contribution labor income for member i of cohort x at age a, $\mathrm{y}^x_L(a) = \int_i \mathrm{y}^x_{Li}(a)\,\mathrm{d}i$ stands for average pre-contribution labor income of cohort x at age a and $\overline{\mathrm{y}}_{Lt} \equiv \frac{\int_A^R N^{t-a}\mathrm{y}^{t-a}_L(a)\,\mathrm{d}a}{\int_A^R N^{t-a}(a)\,\mathrm{d}a}$ denotes average pre-contribution labor income per adult *worker* at time t. The coefficient ι_i is an individual proportionality parameter that is defined as the fraction of individual i's lifetime income to the average lifetime income of his or her cohort, i.e. $\iota_i = \frac{\int_A^R \mathrm{y}^x_{Li}(a)\,\mathrm{d}a}{\int_A^R \mathrm{y}^x_L(a)\,\mathrm{d}a}$.[30] Note that for constant cohort sizes $N^x = N$ average labor income reduces to $\overline{\mathrm{y}}_{Lt} = \frac{\int_A^R \mathrm{y}^{t-a}_L(a)\,\mathrm{d}a}{R-A}$. In the paper I abstract from a seniority wage structure and assume that in each period wages are identical across age groups. In this case it holds that $\mathrm{y}^{t-a}_{Li}(a) = \mathrm{y}_{Lti}$, $\mathrm{y}^{t-a}_L(a) = \mathrm{y}_{Lt}$ and thus $\overline{\mathrm{y}}_{Lt} = \mathrm{y}_{Lt}$ for $a \in [A,R[$. In the absence of seniority wages it also holds that the income position of types is constant over the life-cycle, i.e. $\iota_i = \frac{\int_A^R \mathrm{y}^x_{Li}(a)\,\mathrm{d}a}{\int_A^R \mathrm{y}^x_L(a)\,\mathrm{d}a} = \frac{\mathrm{y}^x_{Li}(a)}{\mathrm{y}^x_L(a)}, \forall a$. One can then write old-age (augmented) income as $y^x_{Li}(a) = \rho(1-\tau_\rho)\mathrm{y}^x_{Li}(a)$ since $\iota_i\overline{\mathrm{y}}_{L,x+a} = \iota_i\mathrm{y}_{L,x+a} = \iota_i\mathrm{y}^x_L(a) = \mathrm{y}^x_{Li}(a)$. It thus also follows that one can transform expression (48) into the time-dependent formulation (22) of the paper.

For the calculation of the equilibrium contribution rate τ^*_ρ it is useful to elaborate on equation (20). In particular, the sizes of the young, old and total populations are given by:

$$\begin{aligned}
N^y_t &= \int_A^R N_t(a)\,\mathrm{d}a = \int_A^R e^{n(t-a)}\,\mathrm{d}a = e^{nt}\frac{e^{-nA}-e^{-nR}}{n}, \\
N^o_t &= \int_R^D N_t(a)\,\mathrm{d}a = \int_R^D e^{n(t-a)}\,\mathrm{d}a = e^{nt}\frac{e^{-nR}-e^{-nD}}{n}, \\
N_t &= N^y_t + N^o_t = \int_A^D N_t(a)\,\mathrm{d}a = \int_A^D e^{n(t-a)}\,\mathrm{d}a = e^{nt}\frac{e^{-nA}-e^{-nD}}{n}.
\end{aligned} \tag{49}$$

The equilibrium contribution rate τ^*_ρ for the PAYG pension system is defined as the rate

[30] This is a short-cut. A specification like this is, however, implied, e.g., by the Swedish Notional Defined Contribution system or by a point system as it is common in a number of countries (Germany, France, Austria).

that balances total revenues and total expenditures which can now be written as:

$$\begin{aligned} Rev_t &= \int_A^R N^{t-a}(a)\tau_\rho \mathrm{y}_L^{t-a}(a)\,\mathrm{d}a = \tau_\rho \mathrm{y}_{Lt}\int_A^R N^{t-a}(a)\,\mathrm{d}a = \tau_\rho \mathrm{y}_{Lt}N_t^y, \\ Exp_t &= \int_R^D N^{t-a}(a)\rho(1-\tau_\rho)\mathrm{y}_L^{t-a}(a)\,\mathrm{d}a = \rho(1-\tau_\rho)\mathrm{y}_{Lt}\int_R^D N^{t-a}(a)\,\mathrm{d}a \\ &= \rho(1-\tau_\rho)\mathrm{y}_{Lt}N_t^o, \end{aligned}$$

using the absence of seniority wages ($\mathrm{y}_L^{t-a}(a) = \mathrm{y}_{Lt}$) and the definitions of N_t^y and N_t^o in equation (49). For constant cohort sizes $N^x = N$ this reduces to $Rev_t = \tau_\rho\,(R-A)\,N\mathrm{y}_{Lt}$ and $Exp_t = \rho(1-\tau_\rho)(D-R)N\mathrm{y}_{Lt}$. A constantly balanced PAYG system requires that $Rev_t = Exp_t$ or $\tau_\rho N_t^y = \rho(1-\tau_\rho)N_t^o$. This can be solved for $\tau_\rho^* = \frac{\rho N_t^o}{N_t^y+\rho N_t^o}$ as stated in equation (23).

One can use the definition of augmented income in (22) and the formula for the balancing contribution rate in (23) to derive the relation between y_{Lt} (average *per worker* pre-contribution-rate labor income) and y_{Lt} (average *per capita* "augmented" labor income). It comes out as:

$$y_{Lt} = \mathrm{y}_{Lt}\frac{N_t^y}{N_t}.$$

In order to see this first note that (augmented) per adult income can also be written as:

$$y_{Lt} = \frac{Y_{Lt}}{N_t} = \frac{\int_A^D Y_L^{t-a}(a)\,\mathrm{d}a}{\int_A^D N^{t-a}(a)\,\mathrm{d}a} = \frac{\int_A^D y_L^{t-a}(a)N^{t-a}(a)\,\mathrm{d}a}{\int_A^D N^{t-a}(a)\,\mathrm{d}a}. \tag{50}$$

Put differently, average (per adult) labor income y_{Lt} is the cohort-size-weighted average of the different age-specific average incomes $y_L^{t-a}(a)$. Using the specification in (22) one can rewrite equation (50) as $y_{Lt} = \frac{\mathrm{y}_{Lt}(1-\tau_\rho)\left(\int_A^R N^{t-a}(a)\,\mathrm{d}a+\rho\int_R^D N^{t-a}(a)\,\mathrm{d}a\right)}{N_t} = \mathrm{y}_{Lt}(1-\tau_\rho)\frac{N_t^y+\rho N_t^o}{N_t}$. Noting that $(1-\tau_\rho) = \frac{N_t^y}{N_t^y+\rho N_t^o}$ the result follows.

A.3 The warm-glow model

In this part of the appendix I provide additional derivations for the benchmark warm-glow (WG) model. The specification and the main results have already been summarized in sections 3 and 4 of the paper.

A.3.1 Budget constraint

At the beginning of adulthood at age A agents start with no financial assets, i.e. $w^x_{Fi}(A) = 0$. From then on they earn a net (augmented) labor income $(1-\tau_y)y^x_{Li}(a)$ (see (22)) where τ_y is a labor income tax rate (in addition to the PAYG contribution rate τ_ρ) and they also collect the return $r^x(a)$ on their accumulated financial wealth $w^x_{Fi}(a)$. It is assumed that all households invest their financial wealth in a fund that holds all available assets of the economy and the rate of return $r^x(a)$ corresponds to the average return of this portfolio (see equation (17)). The period budget constraint thus can be written as:

$$\frac{\mathrm{d}w^x_{Fi}(a)}{\mathrm{d}a} = r^x(a)w^x_{Fi}(a) + (1-\tau_y)y^x_{Li}(a) - \varepsilon^x_i(a) \tag{51}$$

for all periods $a \in [A, D[$ except the bequest period I in which there is an additional income stream b^x_{Fi}. Note that the outright owners also get the family house as an additional bequest at age A. This, however, is not a disposable property (since it has to be passed on to the next generation) and is thus not included in the expression above. The available lifetime resources $\widetilde{y}^x_i$ are given by the total of the discounted stream of net labor income (both during active life and during the pension) $\widetilde{y}^x_{Li}$ and the discounted value of the bequest received at age I. Assuming a constant interest rate r these magnitudes can be written as:

$$\widetilde{y}^x_i = \widetilde{y}^x_{Li} + e^{-r(I-A)}b^x_{Fi} \tag{52}$$

where:

$$\begin{aligned} \widetilde{y}^x_{Li} &= (1-\tau_y)\int_A^D y^x_{Li}(a)e^{-r(a-A)}\,\mathrm{d}a \qquad (53) \\ &= (1-\tau)\mathrm{y}^x_{Li}(A)\left(\int_A^R e^{-(r-g)(a-A)}\,\mathrm{d}a + \rho\int_R^D e^{-(r-g)(a-A)}\,\mathrm{d}a\right), \end{aligned}$$

where I define the total tax rate τ as:

$$(1-\tau) \equiv (1-\tau_y)(1-\tau_\rho). \tag{54}$$

Total lifetime expenditures discounted to age A, on the other hand, are given by:

$$\widetilde{\varepsilon}^x_i = \int_A^D \varepsilon^x_i(a)e^{-r(a-A)}\,\mathrm{d}a. \tag{55}$$

The linear differential equation (51) can be solved for the terminal asset level $w^x_{Fi}(D)$ as:

$$w^x_{Fi}(D) = \int_A^D \left((1-\tau_y)y^x_{Li}(a) - \varepsilon^x_i(a)\right) e^{r(D-a)}\,\mathrm{d}a + b^x_{Fi}e^{r(D-I)}. \tag{56}$$

Dividing both sides by $e^{r(D-A)}$ leads to:

$$\begin{aligned} w^x_{Fi}(D)e^{-r(D-A)} &= \int_A^D \left((1-\tau_y)y^x_{Li}(a) - \varepsilon^x_i(a)\right) e^{-r(a-A)}\,\mathrm{d}a + b^x_{Fi}e^{-r(I-A)} \\ &= \widetilde{y}^x_{Li} + b^x_{Fi}e^{-r(I-A)} - \widetilde{\varepsilon}^x_i = \widetilde{y}^x_i - \widetilde{\varepsilon}^x_i. \end{aligned} \tag{57}$$

The present value of terminal financial assets equals the present value of lifetime income plus the present value of the bequest that is received at age I minus the present value of lifetime expenditures. Note that also for the normal owners it is assumed that they sell their owner-occupied home over the course of their life (e.g. in the form of a reverse mortgage) such that they only hold financial wealth at the moment of death.

For later reference I also define a number of coefficients that allow for a more concise expression of the lifetime values. In particular, one can write equations (52), (53) and (57) as $\widetilde{y}^x_i = \widetilde{y}^x_{Li} + \varphi_{gI}b^x_{Fi}$, $\widetilde{y}^x_{Li} = (1-\tau)\mathrm{y}^x_{Li}(A)\left(\varphi_{gy} + \rho\varphi_{go}\right)$ and $w^x_{Fi}(D)\varphi_{gD} = \widetilde{y}^x_{Li} + b^x_{Fi}\varphi_{gI} - \widetilde{\varepsilon}^x_i$ where:

$$\begin{aligned} \varphi_{gy} &\equiv \int_A^R e^{-(r-g)(a-A)}\,\mathrm{d}a, \varphi_{go} \equiv \int_R^D e^{-(r-g)(a-A)}\,\mathrm{d}a, \\ \varphi_{gI} &\equiv e^{-(r-g)(I-A)}, \varphi_{gD} \equiv e^{-(r-g)(D-A)}. \end{aligned} \tag{58}$$

A.3.2 Household savings

In the paper I focus on the one variant of the model in which households do not reckon with the receipt of a bequest (or—equivalently—they cannot borrow against it). This is also the variant that was emphasized as the most plausible one in Piketty (2010). In this appendix I will, however, first focus on the more straightforward and somewhat simpler variant where households are able to borrow before turning to the case discussed in the paper.

Case with borrowing As a first step one can insert the optimal intratemporal values for $c^x_i(a)$ and $h^x_i(a)$ into (24) and derive the indirect utility function given by $u^x_i(a) =$

$\frac{\varepsilon_i^x(a)}{\left(\frac{P_s^x(a)}{\eta^j}\right)^\gamma}$. This can be inserted into (25) to derive lifetime consumption utility as:

$$U_i^x(A) = \left\{ \int_A^D \frac{e^{-\theta(a-A)}}{\int_A^D e^{-\theta(x-A)}\,\mathrm{d}x} \left(\frac{\varepsilon_i^x(a)}{\left(\frac{P_s^x(a)}{\eta^j}\right)^\gamma} \right)^{1-\sigma} \mathrm{d}a \right\}^{\frac{1}{1-\sigma}}. \tag{59}$$

Using standard methods of dynamic optimization one can derive that along an optimal path total expenditures grow at rate g_ε, i.e. $\varepsilon_i^x(a) = \varepsilon_i^x(A)e^{g_\varepsilon(a-A)}$ with $g_\varepsilon = g_\varepsilon^*$ and g_ε^* as defined in equation (28) and here repeated:

$$g_\varepsilon^* = \frac{r-\theta}{\sigma} + \gamma\frac{\sigma-1}{\sigma}\widetilde{g}. \tag{28}$$

For the derivation of (28) one can set up the Hamiltonian (for a similar derivation, see Grossmann, Larin & Steger 2021):

$$\mathcal{H}(a) = \left(\frac{\varepsilon_i^x(a)}{\left(\frac{P_s^x(a)}{\eta^j}\right)^\gamma} \right)^{1-\sigma} + \lambda(a)\left[rw_{Fi}^x(a) + (1-\tau_y)y_{Li}^x(a) - \varepsilon_i^x(a)\right],$$

where $\lambda(a)$ is the adjoint variable associated with the dynamic budget constraint. The first order conditions are:

$$\begin{aligned} \frac{\partial\mathcal{H}(a)}{\partial\varepsilon_i^x(a)} &= (1-\sigma)\left(\frac{\varepsilon_i^x(a)}{\left(\frac{P_s^x(a)}{\eta^j}\right)^\gamma} \right)^{1-\sigma} \left(\varepsilon_i^x(a)\right)^{-1} - \lambda(a) = 0, \\ -\frac{\dot{\lambda}(a)}{\lambda(a)} &= r-\theta. \end{aligned}$$

Taking the logarithm of the first FOC in order to calculate $\frac{\dot{\lambda}(a)}{\lambda(a)}$ leads to:

$$\frac{\dot{\lambda}(a)}{\lambda(a)} = (-\sigma)\frac{\dot{\varepsilon}_i^x(a)}{\varepsilon_i^x(a)} - (1-\sigma)\,\gamma\frac{\dot{P}_s^x(a)}{P_s^x(a)}.$$

Setting this equal to $(-r+\theta)$ gives:

$$\sigma \frac{\dot{\varepsilon}_i^x(a)}{\varepsilon_i^x(a)} + (1-\sigma)\,\gamma \frac{\dot{P}_s^x(a)}{P_s^x(a)} = r - \theta \longrightarrow$$
$$\frac{\dot{\varepsilon}_i^x(a)}{\varepsilon_i^x(a)} = \frac{r-\theta}{\sigma} + \gamma \frac{\sigma - 1}{\sigma} \frac{\dot{P_s^x}(a)}{P_s^x(a)} = \frac{r-\theta}{\sigma} + \gamma \frac{\sigma-1}{\sigma} \widetilde{g} = g_\varepsilon.$$

The expression for g_ε differs from the usual optimal growth rate in dynamic consumption models due to the fact that houses are in fixed supply and the equilibrium growth of house prices $\widetilde{g}$ might influence optimal expenditure growth. For $\gamma = 0$ or $\sigma = 1$ one gets the standard formula $g_\varepsilon = \frac{r-\theta}{\sigma}$. A crucial special case corresponds to a situation where total expenditures grow at the same rate as incomes, i.e. $g_\varepsilon = g$. For the latter case which is often used in the paper one has to assume that θ and σ adjust such that the equality holds.

Lifetime expenditures are now given by

$$\widetilde{\varepsilon}_i^x = \int_A^D \varepsilon_i^x(a) e^{-r(a-A)}\,\mathrm{d}a = \int_A^D \varepsilon_i^x(A) e^{-(r-g_\varepsilon)(a-A)}\,\mathrm{d}a.$$

This can be used to express the initial expenditure level in terms of lifetime expenditures as:

$$\varepsilon_i^x(A) = \frac{\widetilde{\varepsilon}_i^x}{\int_A^D e^{-(r-g_\varepsilon)(a-A)}\,\mathrm{d}a} = \frac{\widetilde{\varepsilon}_i^x}{\varphi_{g_\varepsilon}} \tag{60}$$

where:

$$\varphi_{g_\varepsilon} \equiv \int_A^D e^{-(r-g_\varepsilon)(a-A)}\,\mathrm{d}a. \tag{61}$$

Using $\varepsilon_i^x(a) = \varepsilon_i^x(A) e^{g_\varepsilon(a-A)}$ it thus follows:

$$\varepsilon_i^x(a) = \frac{\widetilde{\varepsilon}_i^x}{\varphi_{g_\varepsilon}} e^{g_\varepsilon(a-A)}. \tag{62}$$

The latter expression can be inserted into the utility function (59) to give:

$$\begin{aligned} U_i^x(A) = & \left\{ \int_A^D \frac{e^{-\theta(a-A)}}{\int_A^D e^{-\theta(x-A)}\,\mathrm{d}x} \left(\frac{\frac{\widetilde{\varepsilon}_i^x e^{g_\varepsilon(a-A)}}{\varphi_{g_\varepsilon}}}{\left(\frac{P_s^x(a)}{\eta^j}\right)^\gamma} \right)^{1-\sigma} \mathrm{d}a \right\}^{\frac{1}{1-\sigma}} \\ = & \widetilde{\varepsilon}_i^x \left\{ \int_A^D \frac{e^{-\theta(a-A)}}{\int_A^D e^{-\theta(x-A)}\,\mathrm{d}x} \left(\frac{\frac{e^{g_\varepsilon(a-A)}}{\varphi_{g_\varepsilon}}}{\left(\frac{P_s^x(a)}{\eta^j}\right)^\gamma} \right)^{1-\sigma} \mathrm{d}a \right\}^{\frac{1}{1-\sigma}}. \end{aligned} \tag{63}$$

In order to solve for the intergenerational equilibrium one can use $U_i^x(A)$ in the overall utility function V_i^x (see (26)) and maximize this with respect to $\widetilde{\varepsilon}_i^x$ and $w_{Fi}^x(D)$ and subject to the budget constraint (57). This leads to:

$$\widetilde{\varepsilon}_i^x = (1 - s_B^j)\widetilde{y}_i^x, \tag{64}$$

$$w_{Fi}^x(D) = s_B^j e^{r(D-A)}\widetilde{y}_i^x. \tag{65}$$

One can insert $\widetilde{y}_i^x = \widetilde{y}_{Li}^x + e^{-r(I-A)}b_{Fi}^x$ (from (52)) into equation (64) and then insert both into (62) to derive that:

$$\begin{aligned} \varepsilon_i^x(a) &= \frac{\widetilde{\varepsilon}_i^x}{\varphi_{g_\varepsilon}} e^{g_\varepsilon(a-A)} = \frac{1 - s_B^j}{\varphi_{g_\varepsilon}} e^{g_\varepsilon(a-A)}\widetilde{y}_i^x \\ &= \frac{1 - s_B^j}{\varphi_{g_\varepsilon}} e^{g_\varepsilon(a-A)} \left(\widetilde{y}_{Li}^x + e^{-r(I-A)}b_{Fi}^x\right) \\ &= \varepsilon_{Li}^j(a) + \varepsilon_{Bi}^j(a), \end{aligned} \tag{66}$$

where $\varepsilon_{Li}^x(a) \equiv \frac{1-s_B^j}{\varphi_{g_\varepsilon}} e^{g_\varepsilon(a-A)}\widetilde{y}_{Li}^x$ represents the part of total expenditures that are due to labor income while $\varepsilon_{Bi}^x(a) \equiv \frac{1-s_B^j}{\varphi_{g_\varepsilon}} e^{g_\varepsilon(a-A)}e^{-r(I-A)}b_{Fi}^x$ stands for the part that is due to the bequest.

One can define a (possibly age-specific) savings rate $s_L^j(a)$ from net labor income as $\varepsilon_{Li}^x(a) = (1 - s_L^j(a))(1 - \tau_y)y_{Li}^x(a)$. Looking at above one sees that:

$$\begin{aligned} (1 - s_L^j(a)) &= \frac{1 - s_B^j}{\varphi_{g_\varepsilon}}(\varphi_{gy} + \rho\varphi_{go})e^{(g_\varepsilon - g)(a-A)} \text{ for } a \in [A, R[\\ (1 - s_L^j(a)) &= \frac{1 - s_B^j}{\rho\varphi_{g_\varepsilon}}(\varphi_{gy} + \rho\varphi_{go})e^{(g_\varepsilon - g)(a-A)} \text{ for } a \in [R, D] \end{aligned} \tag{67}$$

where I use $\widetilde{y}_{Li}^x = (1-\tau)\mathrm{y}_{Li}^x(A)\,(\varphi_{gy} + \rho\varphi_{go}) = (1-\tau_y)y_{Li}^x(A)\,(\varphi_{gy} + \rho\varphi_{go})$. Note that for

$g_\varepsilon = g$ the savings rate is constant (at $s^j_L(a) = s^j_{Ly} = \frac{1-s^j_B}{\varphi_{g_\varepsilon}}(\varphi_{gy} + \rho\varphi_{go})$ for $a \in [A, R[$ and $s^j_L(a) = s^j_{Lo} = \frac{s^j_{Ly}}{\rho}$ for $a \in [R, D]$ where only for $\rho = 1$ it holds that $s^j_{Ly} = s^j_{Lo}$). For a general g_ε I define $s^j_L \equiv s^j_L(A) = s^j_{Ly}$ and this is the expression to which equation (27) in the text refers:

$$\begin{aligned}(1 - s^j_L) &\equiv \frac{1 - s^j_B}{\varphi_{g_\varepsilon}} \frac{\widetilde{y}^x_{Li}}{(1-\tau)\mathrm{y}^x_{Li}(A)} = \frac{1 - s^j_B}{\varphi_{g_\varepsilon}}(\varphi_{gy} + \rho\varphi_{go}) \\ &= (1 - s^j_B)\frac{r - g_\varepsilon}{r - g} \frac{e^{(r-g)(D-R)}\left(e^{(r-g)(R-A)} - 1\right) + \rho\left(e^{(r-g)(D-R)} - 1\right)}{e^{(r-g)(D-A)} - e^{(g_\varepsilon - g)(D-A)}}, \quad (68)\end{aligned}$$

where I use the definitions for $\widetilde{y}^x_{Li}$ (equation (53)) in the first line and the ones for φ_{g_ε}, φ_{gy} and φ_{go} from above in the second line. Note that for $\rho = 1$ one gets that:

$$\begin{aligned}(1 - s^j_L) &= (1 - s^j_B)\frac{r - g_\varepsilon}{r - g} \frac{e^{(r-g)(D-A)} - 1}{e^{(r-g)(D-A)} + e^{(g_\varepsilon - g)(D-A)}} \\ &= (1 - s^j_B)\frac{r - g_\varepsilon}{r - g} \frac{1 - e^{-(r-g)(D_A)}}{1 - e^{-(r-g_\varepsilon)(D_A)}}, \quad (69)\end{aligned}$$

which is the same expression as in Piketty (2010, p.133).

The equilibrium value for the bequest can be calculated from equation (65) and the equilibrium condition $w^x_{Fi}(D) = b^{x+(D-I)}_{Fi} e^{n(D-I)}$ from equation (29) (this will be shown in more detail for the non-borrowing case below). This leads to:

$$b^x_{Fi} = s^j_B \frac{e^{(r-g)(D-A)} e^{g(I-A)}}{e^{n(D-I)} - s^j_B e^{(r-g)(D-I)}} \widetilde{y}^x_{Li} \quad (70)$$

which is the same value as (30) (which was derived for the case without borrowing). Note that the equilibrium condition (29) implicitly assumes that there is perfect heritability of the productivity type.[31]

Expression (70) can be compared to equation (E.13) on p.134 in Piketty (2010).

Case without borrowing I now assume, as in parts (ii) and (iii) of section E5 in Piketty (2010), that households cannot borrow against their future inheritance. In particular, it is assumed that until age I they behave as if they were not going to receive any inheritance. At age I they then become aware of the bequest and they adapt their plans accordingly. Put differently, they maximize their lifetime utility function $V^x_i(U^x_i, w^x_i(D))$

[31] And also identical numbers of offsprings for each type i. One could also assume the existence of a proportional inheritance tax τ_b which would not change the main results.

twice: once at age A (assuming no inheritance) and a second time at age I (now including the just received inheritance). At age A optimal consumption smoothing again implies the same optimal path as in the case with borrowing. In particular,:

$$\varepsilon_i^x(a) = \varepsilon_i^x(A)e^{g_\varepsilon(a-A)}$$

with $g_\varepsilon = \frac{r-\theta}{\sigma} + \gamma\frac{\sigma-1}{\sigma}\widetilde{g}$.

Different to before, however, households now behave at age A as if the present value of their lifetime resources would only consist of labor income: $\widetilde{y}_i^x(A) = \widetilde{y}_{Li}^x(A)$.[32]

The intergenerational maximization at age A now leads to the following expressions:

$$\widetilde{\varepsilon}_{Li}^x(A) = (1 - s_B^j)\widetilde{y}_{Li}^x(A), \tag{71}$$

$$w_{Li}^x(D) = s_B^j e^{r(D-A)}\widetilde{y}_{Li}^x(A). \tag{72}$$

It thus follows that for $a \in [A, I[$ one has that

$$\varepsilon_i^x(a) = \varepsilon_{Li}^x(a) = (1 - s_L^j)(1 - \tau)\mathrm{y}_{Li}^x(A)e^{g_\varepsilon(a-A)} = (1 - s_L^j)(1 - \tau)\mathrm{y}_{Li}^x(a)e^{(g_\varepsilon - g)(a-A)}.$$

as shown in equation (27). For this period until age I it holds that $\varepsilon_{Bi}^x(a) = 0$.

This changes at age I when they receive a bequest and they reoptimize. As stated in the paper, I follow here Piketty and assume that households leave the part of their expenditures that are financed from labor income completely unchanged and that they only add an extra element financed from the newly received bequest. This means that $\varepsilon_{Li}^x(a)$ is the same over the entire period $a \in [A, D]$ and given by $\varepsilon_{Li}^x(a) = (1 - s_L^j)(1 - \tau)\mathrm{y}_{Li}^x(A)e^{g_\varepsilon(a-A)}$ as specified in equation (27). For the expenditures from the bequest it holds that: $\varepsilon_{Bi}^x(a) = \varepsilon_{Bi}^x(I)e^{g_\varepsilon(a-I)}$ where $\varepsilon_{Bi}^x(I) = \frac{\widetilde{\varepsilon}_{Bi}^x}{\int_I^D e^{-(r-g_\varepsilon)(a-I)}\,\mathrm{d}a}$ (which follows from $\widetilde{\varepsilon}_{Bi}^x = \int_I^D \varepsilon_{Bi}^x(a)e^{-r(a-I)}\,\mathrm{d}a = \int_I^D \varepsilon_{Bi}^x(I)e^{-(r-g_\varepsilon)(a-I)}\,\mathrm{d}a$).[33]

The optimal split of the received bequest in own expenditures and planned bequest comes out as:

$$\widetilde{\varepsilon}_{Bi}^x = (1 - s_B^j)b_{Fi}^x, \tag{73}$$

[32]Note that I now add an age term in brackets to the present value variables (denoted by a tilde) in order to indicate at which age these present values are assessed. It holds, e.g., that $\widetilde{y}_i^x(A) \neq \widetilde{y}_i^x(I)$.

[33]As an alternative one could also assume that households re-maximize, thereby taking the new budget constraint at age I into account with $\widetilde{y}_i^x(I) = \widetilde{y}_{Li}^x(I) + b_{Fi}^x + w_{Fi}^x(I)$ where $\widetilde{y}_{Li}^x(I) = \int_I^D(1 - \tau_y)y_{Li}^x(a)e^{-r(a-A)}\,\mathrm{d}a$ and $w_{Fi}^x(I)$ is the accumulated wealth at age I. For the sake of simplicity and continuity with the assumptions of Piketty (2010) I stick to the described model.

$$w^x_{Bi}(D) = \frac{s^j_B}{e^{-r(D-I)}} b^x_{Fi}. \tag{74}$$

The overall solution for $\varepsilon^x_{Bi}(a)$ for the model without borrowing is then given by $\varepsilon^x_{Bi}(a) = \frac{1-s^j_B}{\int_I^D e^{-(r-g_\varepsilon)(a-I)}\,\mathrm{d}a} b^x_{Fi} e^{g_\varepsilon(a-I)} = (1-\breve{s}^j_B) b^x_{Fi} e^{g_\varepsilon(a-I)}$ where I define:

$$1-\breve{s}^j_B \equiv \frac{1-s^j_B}{\int_I^D e^{-(r-g_\varepsilon)(a-I)}\,\mathrm{d}a}. \tag{75}$$

Overall the solution for $\varepsilon^x_i(a)$ has been stated as (27) using these definitions of $1-s^j_L$ and $1-\breve{s}^j_B$ from (68) and (75).

Alternatively, however, one could also insert for the equilibrium value of b^x_{Fi} (from (30)) to write expression (27) in a slightly different form:

$$\varepsilon^x_i(a) = \begin{cases} (1-\tau)\mathrm{y}^x_{Li}(A)(1-s^j_L)e^{g_\varepsilon(a-A)} & \text{for } a \in [A, I[\\ (1-\tau)\mathrm{y}^x_{Li}(A)\left[(1-s^j_L)e^{g_\varepsilon(a-A)} + (1-\widetilde{s}^j_B)e^{g_\varepsilon(a-I)}\right] & \text{for } a \in [I, D] \end{cases} \tag{76}$$

where now:

$$\begin{aligned} (1-\widetilde{s}^j_B) &\equiv \frac{1-s^j_B}{\int_I^D e^{-(r-g_\varepsilon)(a-I)}\,\mathrm{d}a} \frac{b^x_{Fi}}{(1-\tau)\mathrm{y}^x_{Li}(A)} \\ &= \frac{s^j_B(1-s^j_B)}{\int_I^D e^{-(r-g_\varepsilon)(a-I)}\,\mathrm{d}a} \frac{e^{(r-g)(D-A)}e^{g(I-A)}}{e^{n(D-I)} - s^j_B e^{(r-g)(D-I)}} \frac{\widetilde{y}^x_{Li}}{(1-\tau)\mathrm{y}^x_{Li}(A)} \\ &= (1-s^j_B)s^j_B \frac{r-g_\varepsilon}{r-g} \frac{e^{(r-g)(D-R)}\left(e^{(r-g)(R-A)}-1\right) + \rho\left(e^{(r-g)(D-R)}-1\right)}{\left(1-e^{-(r-g_\varepsilon)(D-I)}\right)\left(e^{n(D-I)} - s^j_B e^{(r-g)(D-I)}\right)} \\ &= (1-s^j_L)s^j_B \frac{e^{(r-g)(D-A)} - e^{(g_\varepsilon-g)(D-A)}}{\left(1-e^{-(r-g_\varepsilon)(D-I)}\right)\left(e^{n(D-I)} - s^j_B e^{(r-g)(D-I)}\right)}, \end{aligned} \tag{77}$$

using (53) and (30) in lines 3 and 2, respectively.

The equilibrium value of bequest b^x_{Fi} (see equation (30)) can thereby be derived from the wealth at age D. As sketched in the paper, the wealth at age D comes out from the optimal intergenerational choice as:

$$w^x_{Fi}(D) = w^x_{Li}(D) + w^x_{Bi}(D) = s^j_B e^{r(D-A)}\widetilde{y}^x_{Li} + \frac{s^j_B}{e^{-r(D-I)}} b^x_{Fi}.$$

The equilibrium condition on the other hand (involving the fact that one's generation terminal wealth is the next generation's level of bequest) is given by $w^x_{Fi}(D) = b^{x+(D-I)}_{Fi} e^{n(D-I)}$

from equation (29). Setting these two expressions equals to each other and noting that in equilibrium $b_{Fi}^{x+(D-I)} = b_{Fi}^{x} e^{g(D-I)}$ leads to:

$$\begin{aligned} s_B^j e^{r(D-A)} \widetilde{y}_{Li}^x + \frac{s_B^j}{e^{-r(D-I)}} b_{Fi}^x &= b_i^{x+(D-I)} e^{n(D-I)} = b_{Fi}^x e^{(g+n)(D-I)} \rightarrow \\ b_{Fi}^x \left(e^{(g+n)(D-I)} - \frac{s_B^j}{e^{-r(D-I)}} \right) &= s_B^j e^{r(D-A)} \widetilde{y}_{Li}^x. \end{aligned}$$

Assuming that $e^{n(D-I)} - s_B^j e^{(r-g)(D-I)} > 0$ one can derive from this expression that $b_{Fi}^x = s_B^j \frac{e^{(r-g)(D-A)} e^{g(I-A)}}{e^{n(D-I)} - s_B^j e^{(r-g)(D-I)}} \widetilde{y}_{Li}^x$ as stated in equation (30).

A.4 Household wealth

In the next step one can derive the age-specific levels of financial $w_{Fi}^x(a)$. It is again useful to write $w_{Fi}^x(a) = w_{Li}^x(a) + w_{Bi}^x(a)$, where $w_{Li}^x(a)$ refers to the part of wealth that originates in the accumulation of labor income while $w_{Bi}^x(a)$ refers to a second part that is due to the financial bequest. In order to prevent notational clutter I focus again on the steady state with $r_t = r$. The labor-income related part of wealth is given by:

$$w_{Li}^x(a) = \begin{cases} \int_A^a \left[\mathrm{y}_{Li}^x(s)(1-\tau) - \varepsilon_{Li}^x(s) \right] e^{r(a-s)} \, \mathrm{d}s & \text{for } a \in [A, R[\\ \int_A^R \left[\mathrm{y}_{Li}^x(s)(1-\tau) - \varepsilon_{Li}^x(s) \right] e^{r(a-s)} \, \mathrm{d}s & \text{for } a \in [R, D] \\ + \int_R^a \left[\rho \mathrm{y}_{Li}^x(s)(1-\tau) - \varepsilon_{Li}^x(s) \right] e^{r(a-s)} \, \mathrm{d}s, & \end{cases} \tag{78}$$

where $\varepsilon_{Li}^x(s) = (1 - s_L^j)(1-\tau) \mathrm{y}_{Li}^x(A) e^{g_\varepsilon(s-A)}$ (see equation (76)).

The bequest-related part of wealth, on the other hand, is given by:

$$w_{Bi}^x(a) = \begin{cases} 0 & \text{for } a \in [A, I[\\ b_{Fi}^x e^{r(a-I)} - \int_I^a \varepsilon_{Bi}^x(s) e^{r(a-s)} \, \mathrm{d}s & \text{for } a \in [I, D] \end{cases} \tag{79}$$

where $\varepsilon_{Bi}^x(s) = (1-\tau)\mathrm{y}_{Li}^x(A)(1-\widetilde{s}_B^j) e^{g_\varepsilon(s-I)} = \frac{1-s_B^j}{\int_I^D e^{-(r-g_\varepsilon)(a-I)} \, \mathrm{d}a} b_{Fi}^x e^{g_\varepsilon(s-I)}$ (see again equation (76)). The direct owners always have an additional component of wealth given by $w_{Hi}^x(a) = P_h^x(a) h_i^x(a)$ which is not on the market but is nevertheless used to assess their net worth.

It holds that $\mathrm{y}_{Li}^x(s) = \mathrm{y}_{Li}^x(A) e^{g(s-A)}$ and $\varepsilon_{Li}^x(s) = \varepsilon_{Li}^x(A) e^{g_\varepsilon(s-A)}$. Inserting this into

(78) one arrives at:

$$w_{Li}^x(a) = \begin{cases} \mathrm{y}_{Li}^x(A)(1-\tau)e^{g(a-A)} \int_A^a e^{(r-g)(a-s)}\,\mathrm{d}s & \text{for } a \in [A,R[\\ -\varepsilon_{Li}^x(A)e^{g_\varepsilon(a-A)} \int_A^a e^{(r-g_\varepsilon)(a-s)}\,\mathrm{d}s & \\ \mathrm{y}_{Li}^x(A)(1-\tau)e^{g(a-A)} \left(\int_A^R e^{(r-g)(a-s)}\,\mathrm{d}s + \rho\int_R^a e^{(r-g)(a-s)}\,\mathrm{d}s\right) & \text{for } a \in [R,D] \\ -\varepsilon_{Li}^x(A)e^{g_\varepsilon(a-A)} \int_A^a e^{(r-g_\varepsilon)(a-s)}\,\mathrm{d}s. & \end{cases}$$

In order to simplify the further derivation, the model can be written in terms of deviations from a stable growth path. In particular, stationary variables are defined by dividing each variable by the labor income of labor market entrants. This means that for all variables $z_i^x(a)$ except the house prices I define $\widehat{z}_i(a) \equiv \frac{z_i^x(a)}{\mathrm{y}_{L,x+a}(A)} = \frac{z_i^x(a)}{\mathrm{y}_{Lx}(A)e^{g(a-A)}}$ or, equivalently, $\widehat{z}_i(a) \equiv \frac{z_{ti}(a)}{\mathrm{y}_{Lt}(A)}$. Note again that also all stationary variables are potentially group-specific, i.e. they could be written in extended form as $\widehat{z}_i^j(a)$. As discussed in section 2.2, house prices grow at rate $\widetilde{g} = g + n(1-\chi)$ along the balanced growth path. In order to rewrite equations (78) one can use $\mathrm{y}_{Li}^x(s) = \mathrm{y}_{Li}^x(A)e^{g(s-A)} = \mathrm{y}_{Li}^x(A)e^{-g(a-s)}e^{g(a-A)}$, $w_{Li}^x(a) = \widehat{w}_{Li}(a)\mathrm{y}_L^x(A)e^{g(a-A)}$ and $\mathrm{y}_{Li}^x(A) = \mathrm{y}_L^x(A)\widehat{\mathrm{y}}_{Li}(A)$ to write:

$$\widehat{w}_{Li}(a)\mathrm{y}_L^x(A)e^{g(a-A)} = \begin{cases} \mathrm{y}_L^x(A)e^{g(a-A)}\widehat{\mathrm{y}}_{Li}(A)(1-\tau)\int_A^a e^{(r-g)(a-s)}\,\mathrm{d}s & \text{for } a \in [A,R[\\ -\mathrm{y}_L^x(A)e^{g(a-A)}\widehat{\varepsilon}_{Li}(A)e^{(g_\varepsilon-g)(a-A)} \int_A^a e^{(r-g_\varepsilon)(a-s)}\,\mathrm{d}s & \\ \mathrm{y}_L^x(A)e^{g(a-A)}\widehat{\mathrm{y}}_{Li}(A)(1-\tau) & \text{for } a \in [R,D] \\ \left(\int_A^R e^{(r-g)(a-s)}\,\mathrm{d}s + \rho\int_R^a e^{(r-g)(a-s)}\,\mathrm{d}s\right) & \\ -\mathrm{y}_L^x(A)e^{g(a-A)}\widehat{\varepsilon}_{Li}(A)e^{(g_\varepsilon-g)(a-A)} \int_A^a e^{(r-g_\varepsilon)(a-s)}\,\mathrm{d}s. & \end{cases}$$

Dividing both sides by $\mathrm{y}_L^x(A)e^{g(a-A)}$ this leads to.

$$\widehat{w}_{Li}(a) = \begin{cases} \widehat{\mathrm{y}}_{Li}(A)(1-\tau)\int_A^a e^{(r-g)(a-s)}\,\mathrm{d}s & \text{for } a \in [A,R[\\ -\widehat{\varepsilon}_{Li}(A)e^{(g_\varepsilon-g)(a-A)} \int_A^a e^{(r-g_\varepsilon)(a-s)}\,\mathrm{d}s & \\ \widehat{\mathrm{y}}_{Li}(A)(1-\tau)\left(\int_A^R e^{(r-g)(a-s)}\,\mathrm{d}s + \rho\int_R^a e^{(r-g)(a-s)}\,\mathrm{d}s\right) & \text{for } a \in [R,D] \\ -\widehat{\varepsilon}_{Li}(A)e^{(g_\varepsilon-g)(a-A)} \int_A^a e^{(r-g_\varepsilon)(a-s)}\,\mathrm{d}s. & \end{cases} \tag{80}$$

One can insert here for $\widehat{\varepsilon}_{Li}(A) = (1-s_L^j)(1-\tau)\widehat{\mathrm{y}}_{Li}(A)$ to arrive at:

$$\widehat{w}_{Li}(a) = \begin{cases} \widehat{\mathrm{y}}_{Li}(A)\Big[(1-\tau)\int_A^a e^{(r-g)(a-s)}\,\mathrm{d}s & \text{for } a \in [A,R[\\ -(1-s_L^j)(1-\tau)e^{(g_\varepsilon-g)(a-A)}\int_A^a e^{(r-g_\varepsilon)(a-s)}\,\mathrm{d}s\Big] & \\ \widehat{\mathrm{y}}_{Li}(A)\Big[(1-\tau)\int_A^R e^{(r-g)(a-s)}\,\mathrm{d}s + (1-\tau)\rho\int_R^a e^{(r-g)(a-s)}\,\mathrm{d}s & \text{for } a \in [R,D] \\ -(1-s_L^j)(1-\tau)e^{(g_\varepsilon-g)(a-A)}\int_A^a e^{(r-g_\varepsilon)(a-s)}\,\mathrm{d}s\Big]\,. & \end{cases} \tag{81}$$

One can make similar transformations for $w_{Bi}^x(a)$ in equation (79). In particular, it holds also for bequest-related expenditures that they grow at the optimal rate g_ε, i.e. $\varepsilon_{Bi}^x(s) = \varepsilon_{Bi}^x(I)e^{g_\varepsilon(s-I)}$. This leads to:

$$w_{Bi}^x(a) = \begin{cases} 0 & \text{for } a \in [A,I[\\ b_{Fi}^x e^{r(a-I)} - \varepsilon_{Bi}^x(I)e^{g_\varepsilon(a-I)}\int_I^a e^{(r-g_\varepsilon)(a-s)}\,\mathrm{d}s & \text{for } a \in [I,D]. \end{cases}$$

Noting again that $w_{Bi}^x(a) = \widehat{w}_{Bi}(a)\mathrm{y}_L^x(A)e^{g(a-A)}$, $\varepsilon_{Bi}^x(a) = \widehat{\varepsilon}_{Bi}(a)\mathrm{y}_L^x(I)e^{g(a-I)}$ (or $\varepsilon_{Bi}^x(I) = \mathrm{y}_L^x(I)\widehat{\varepsilon}_{Bi}(I)$) and $b_{Fi}^x = \widehat{b}_{Fi}\mathrm{y}_L^x(I)$ one can also write:

$$\widehat{w}_{Bi}(a)\mathrm{y}_L^x(A)e^{g(a-A)} = \begin{cases} 0 & \text{for } a \in [A,I[\\ \mathrm{y}_L^x(A)e^{g(a-A)}e^{(r-g)(a-I)}\widehat{b}_{Fi} & \text{for } a \in [I,D] \\ -\mathrm{y}_L^x(A)e^{g(a-A)}e^{(g_\varepsilon-g)(a-I)}\widehat{\varepsilon}_{Bi}(I)\int_I^a e^{(r-g_\varepsilon)(a-s)}\,\mathrm{d}s. & \end{cases}$$

Dividing both sides by $\mathrm{y}_L^x(A)e^{g(a-A)}$ this leads to.

$$\widehat{w}_{Bi}(a) = \begin{cases} 0 & \text{for } a \in [A,I[\\ e^{(r-g)(a-I)}\widehat{b}_{Fi} - e^{(g_\varepsilon-g)(a-I)}\widehat{\varepsilon}_{Bi}(I)\int_I^a e^{(r-g_\varepsilon)(a-s)}\,\mathrm{d}s. & \text{for } a \in [I,D] \end{cases}$$

Inserting $\widehat{\varepsilon}_{Bi}(I) = \frac{1-s_B^j}{\int_I^D e^{-(r-g_\varepsilon)(a-I)}\,\mathrm{d}a}\widehat{b}_{Fi} = (1-\breve{s}_B)\widehat{b}_{Fi}$ into this expression finally leads to:

$$\widehat{w}_{Bi}(a) = \begin{cases} 0 & \text{for } a \in [A,I[\\ \widehat{b}_i\left(e^{(r-g)(a-I)} - (1-\breve{s}_B)e^{(g_\varepsilon-g)(a-I)}\int_I^a e^{(r-g_\varepsilon)(a-s)}\,\mathrm{d}s\right). & \text{for } a \in [I,D] \end{cases} \tag{82}$$

In a final step one can derive the average values where I reintroduce the group-specific superscript j: $\widehat{w}^j(a) = \int_{i\in j}\widehat{w}_i(a)\,\mathrm{d}i$, $\widehat{w}_L^j(a) = \int_{i\in j}\widehat{w}_{Li}(a)\,\mathrm{d}i$, $\widehat{w}_B^j(a) = \int_{i\in j}\widehat{w}_{Bi}(a)\,\mathrm{d}i$, $\widehat{b}_F^j = \int_{i\in j}\widehat{b}_{Fi}\,\mathrm{d}i$. Note that $\widehat{w}_F^j(a) = \widehat{w}_L^j(a) + \widehat{w}_B^j(a)$. For income it holds that $\int_{i\in j}\widehat{\mathrm{y}}_{Li}(A)\,\mathrm{d}i =$

$\int_{i\in j}\frac{\mathrm{y}^x_{Li}(A)}{\mathrm{y}^x_L(A)}\,\mathrm{d}i=\int_{i\in j}\frac{d^j_y\mathrm{y}^x_L(A)}{\mathrm{y}^x_L(A)}\,\mathrm{d}i=d^j_y$.

A.5 Aggregate wealth

It holds that $w^j_{Ft}(a)=\widehat{w}^j_F(a)\mathrm{y}_{Lt}$. This implies that for group j aggregate households' financial savings are given by:

$$W^{j,s}_{Ft}=\int_A^D w^j_{Ft}(a)N^j_t(a)\,\mathrm{d}a=\int_A^D \kappa^j_N e^{n(t-a)}\widehat{w}^j_F(a)\mathrm{y}_{Lt}\,\mathrm{d}a=\kappa^j_N\mathrm{y}_{Lt}\int_A^D e^{n(t-a)}\widehat{w}^j_F(a)\,\mathrm{d}a.$$

The values for $\widehat{w}^j_F(a)$ differ between households because they depend on s^j_B, s^j_L and also on d^j_y (see $\int_{i\in j}\widehat{\mathrm{y}}_{Li}(A)\,\mathrm{d}i=d^j_y$ from the previous section). The aggregate supply of financial wealth is then given by $W^s_{Ft}=\sum_j W^{j,s}_{Ft}$. Expressed as a ratio of aggregate normal output Y_{Nt} this can be written as $\widetilde{\beta}^{j,N}_{Ft}\equiv\frac{W^{j,s}_{Ft}}{Y_{Nt}}=(1-\alpha)\frac{W^{j,s}_{Ft}}{Y_{Lt}}$. I distinguish wealth again by its source and write $W^{j,s}_{Lt}=\kappa^j_N\mathrm{y}_{Lt}\int_A^D e^{n(t-a)}\widehat{w}^j_L(a)\,\mathrm{d}a$ and $W^{j,s}_{Bt}=\kappa^j_N\mathrm{y}_{Lt}\int_A^D e^{n(t-a)}\widehat{w}^j_B(a)\,\mathrm{d}a$ and the corresponding wealth-to-income ratios $\widetilde{\beta}^{j,N}_{Lt}=\frac{W^{j,s}_{Lt}}{Y_{Nt}}$ and $\widetilde{\beta}^{j,N}_{Bt}=\frac{W^{j,s}_{Bt}}{Y_{Nt}}$. Note that $\widetilde{\beta}^{j,N}_{Ft}=\widetilde{\beta}^{j,N}_{Lt}+\widetilde{\beta}^{j,N}_{Bt}$ or—using steady state assumptions—$\widetilde{\beta}^{j,N}_F=\widetilde{\beta}^{j,N}_L+\widetilde{\beta}^{j,N}_B$.

One can now use the formulas for (average) household wealth (81) and (82) to derive closed-form solutions for $\widetilde{\beta}^{j,N}_L$ and $\widetilde{\beta}^{j,N}_B$. After some tedious calculations and transformations they come out as:

$$\begin{aligned}\widetilde{\beta}^{j,N}_L &= \kappa^j_N d^j_y(1-\alpha)(1-\tau)\frac{e^{-nL_o}}{e^{nL_y}-1}\left\{\frac{n\left(e^{(r-g)(L_y+L_o)}-(1-\rho)e^{(r-g)L_o}\right)}{(r-(g+n))(r-g)}\right.\\ &\quad-\frac{\left(e^{n(L_y+L_o)}-(1-\rho)e^{nL_o}\right)}{r-(g+n)}+\frac{\rho}{r-g}-(1-s^j_L)n\times \qquad (83)\\ &\quad\left.\left(\frac{e^{(r-g)(L_y+L_o)}}{(r-(g+n))(r-g_\varepsilon)}+\frac{e^{(g_\varepsilon-g)(L_y+L_o)}}{(r-g_\varepsilon)(n+g-g_\varepsilon)}-\frac{e^{n(L_y+L_o)}}{(r-(g+n))(n+g-g_\varepsilon)}\right)\right\},\\ \widetilde{\beta}^{j,N}_B &= \kappa^j_N(1-\alpha)\widehat{b}^j_F\frac{ne^{-nL_o}}{e^{nL_y}-1}\left\{\frac{e^{(r-g)L_b}-e^{nL_b}}{(r-(g+n)}-\frac{(1-s^j_B)(r-g_\varepsilon)}{1-e^{-(r-g_\varepsilon)L_b}}\times\right. \qquad (84)\\ &\quad\left.\left(\frac{e^{(r-g)L_b}}{(r-(g+n))(r-g_\varepsilon)}+\frac{e^{(g_\varepsilon-g)L_b}}{(n+g-g_\varepsilon)(r-g_\varepsilon)}-\frac{e^{nL_b}}{(r-(g+n))(n+g-g_\varepsilon)}\right)\right\},\end{aligned}$$

where three time intervals are defined as:

$$L_y=R-A,L_b=D-I,L_o=D-R \qquad (85)$$

and where $\widehat{b}_F^j$ can be derived from equation (30) as:

$$\widehat{b}_F^j = s_B^j \frac{e^{(r-g)(D-A)}}{e^{n(D-I)} - s_B^j e^{(r-g)(D-I)}} \widehat{\widetilde{y}}_L^j(A) \tag{86}$$

with $\widehat{\widetilde{y}}_L^j(A) = (1-\tau)(\varphi_{gy} + \rho\varphi_{go})d_y^j$.

One can look at various simplified versions of (83) and (84). For the case where $\rho = 1$, $\tau_y = 0$ and $g = g_\varepsilon$ one gets, e.g., that (after inserting for $(1-\tau_\rho)$ and $(1-s_L^j)$):

$$\widetilde{\beta}_L^{j,N} \;=\; \kappa_N^j d_y^j (1-\alpha) \frac{1}{e^{n(L_y+L_o)} - 1} s_B^j \frac{n\left(e^{(r-g)(L_y+L_o)} - 1\right) - (r-g)\left(e^{n(L_y+L_o)} - 1\right)}{(r-(g+n))(r-g)}.$$

This means that for $\rho = 1$ and $g = g_\varepsilon$ and also $s_B^j = 0$ there is no savings motive left and one gets $\widetilde{\beta}_L^{j,N} = \widetilde{\beta}_B^{j,N} = \widetilde{\beta}_F^{j,N} = 0$. Note that this is not true if $\rho = 1$ and $s_B^j = 0$ but $g \neq g_\varepsilon$ (where it would only hold for $n = 0$).

For $s_B^j = 0$, $\rho = 0$, $\tau_y = 0$ and $r = g = g_\varepsilon = 0$, on the other hand, one gets:

$$\widetilde{\beta}_F^N = \frac{e^{-nL_o}\left(e^{nL_o} L_o \left(e^{nL_y} - 1\right) - L_y\left(e^{nL_o} - 1\right)\right)}{\left(e^{nL_y} - 1\right)(L_y + L_o)n}.$$

For $\lim_{n\to 0}$ one gets that in this case $\frac{\partial \widetilde{\beta}_F^N}{\partial n} = -\frac{L_o(2L_o+L_y)}{12} < 0$. So faster population growth leads to lower wealth. This was called by Modigliani the “Neisser effect”.

For $s_B^j = 0$, $\rho = 0$, $\tau_y = 0$ and $r = g = g_\varepsilon = n = 0$ it holds that $\widetilde{\beta}_F^N = (1-\alpha)\frac{D-R}{2}$ which of course is the famous triangular formula (which is mostly derived for the case where $\alpha = 0$ or where—equivalently—wealth is expressed as a ratio to labor income when the ratio is simply $\frac{D-R}{2}$).

A.6 Additional aggregate magnitudes

A.6.1 Aggregate savings

For the model of section 4 expenditures were given by equation (27) which is here restated in terms of the average age-specific expenditure levels in period t:

$$\varepsilon_t^j(a) = \begin{cases} (1-s_L^j)(1-\tau)\mathrm{y}_{Lt}^j e^{g_\varepsilon(a-A)} & \text{for } a \in [A, I[\\ (1-\tau)\mathrm{y}_{Lt}^j(1-s_L^j)e^{g_\varepsilon(a-A)} + (1-\breve{s}_B^j)b_{Fti}^j e^{g_\varepsilon(a-I)}) & \text{for } a \in [I, D]. \end{cases} \tag{87}$$

Using (87) in $\mathcal{E}_t^j = \int_A^D \varepsilon_t^j(a) N_t^j(a)\,\mathrm{d}a$ one can derive (after some transformations) that:

$$\begin{aligned} \mathcal{E}_t^j &= \kappa_N^j d_y^j Y_{Nt}(1-\alpha)(1-\tau)\frac{e^{-nL_o}n}{e^{nL_y}-1} \times \\ & \frac{(1-s_L^j)\left(e^{n(L_y+L_o)} - e^{(g_\varepsilon-g)(L_y+L_o)}\right) + (1-\widetilde{s}_B^j)\left(e^{nL_b} - e^{(g_\varepsilon-g)L_b}\right)}{n+g-g_\varepsilon}, \end{aligned}$$

which has already been stated as equation (34) in the paper. The terms $(1-s_L^j)$ and $(1-\widetilde{s}_B^j)$ are defined in appendix A.3 as equations (68) and (77).

Savings rates are often defined in relation to different income concepts. In general one can say that for some arbitrary income concept Z_t the associated savings rate is defined as $\overline{s}^z = \frac{Z_t - \mathcal{E}_t}{Z_t} = 1 - \frac{\mathcal{E}_t}{Z_t}$. Noting that from (35) one can write $\frac{\mathcal{E}_t}{GDP_t} = 1 - \overline{s}$ one can thus calculate these adjusted savings rate directly as $\overline{s}^z = 1 - (1-\overline{s})\frac{GDP_t}{Z_t}$ (which only requires a knowledge of the gross savings rate $\overline{s}$ and the relative income magnitude $\frac{GDP_t}{Z_t}$). In the paper I have shown this for the net-savings rate in equation (36), i.e. $\overline{s}^{net} = 1 - (1-\overline{s})\,\frac{GDP_t}{NDP_t}$. The formula implies that the ratio of the "expenditure rates" is inversely proportional to the ratio of the income concepts, i.e. $\frac{1-\overline{s}}{1-\overline{s}^{net}} = \frac{NDP_t}{GDP_t}$.

The "national-account gross savings rate", on the other hand, was reported in equation (37) as: $\overline{s}^{NA} = 1 - \frac{GDP_t}{GDP_t^{NA}}\,(1-\overline{s})$. It should be noted that in this specification of GDP_t^{NA} and the associated savings rate it is assumed that no capital gains from housing assets are redistributed as asset income (since then they would show up as disposable income of the households). This might not be true for all housing assets where especially commercial real estate firms might also redistribute capital gains as dividends. On the other hand, also some gains from the ownership of physical capital are sometimes not directly redistributed to the investors but used to buy back shares or held as corporate savings (Chen et al. 2017). This is a complicated issue and I will stick in this paper to definition (37).

A.6.2 Equilibrium house prices

As stated in section 4.2.3 the equilibrium rents come out as $P^r_{st} = \frac{\gamma}{\overline{H}^r_t}\mathcal{E}^r_t$ and $P^o_{st} = \frac{\gamma}{\overline{H}^{om}_t}\mathcal{E}^{om}_t$. One can insert for $\mathcal{E}^j_t$ from (34) and thus write that:

$$\begin{aligned} P^r_{st} &= \kappa^r_N d^r_y \frac{Y_{Nt}}{\overline{H}^r_t}\gamma(1-\alpha)(1-\tau_\rho)\frac{e^{-nL_o}n}{e^{nL_y}-1}\times \\ &\quad \frac{(1-s^r_L)\left(e^{n(L_y+L_o)}-e^{(g_\varepsilon-g)(L_y+L_o)}\right)+(1-\widetilde{s}^r_B)\left(e^{nL_b}-e^{(g_\varepsilon-g)L_b}\right)}{n+g-g_\varepsilon}, \quad (88)\\ P^o_{st} &= \kappa^{om}_N d^{om}_y \frac{Y_{Nt}}{\overline{H}^{om}_t}\gamma(1-\alpha)(1-\tau_\rho)\frac{e^{-nL_o}n}{e^{nL_y}-1}\times \\ &\quad \frac{(1-s^{om}_L)\left(e^{n(L_y+L_o)}-e^{(g_\varepsilon-g)(L_y+L_o)}\right)+(1-\widetilde{s}^{om}_B)\left(e^{nL_b}-e^{(g_\varepsilon-g)L_b}\right)}{n+g-g_\varepsilon}. \quad (89)\end{aligned}$$

Note that a smaller housing supply does not have an effect on the fractions $\frac{P^r_{st}\overline{H}^r_t}{Y_{Nt}}$ and $\frac{P^o_{st}\overline{H}^{om}_t}{Y_{Nt}}$ since the smaller supply will only increases prices without having an impact on the share.

Due to the assumption that renters and owners with mortgage have the same income and the same bequest motive, i.e. $d^r_y = d^{om}_y = d_y$ and $s^r_B = s^{om}_B = s_B$ equation (38) with $\frac{P^r_{st}}{P^o_{st}} = \frac{\kappa^r_N}{\kappa^{om}_N}\frac{\kappa^{om}_H}{\kappa^r_H}$ follows.

A.6.3 Aggregate bequest

The inheritance flow (from financial bequests) b_{yFt} is defined as:

$$b^N_{yFt} = \frac{B_{Ft}}{Y_{Nt}}, \quad (90)$$

where B_{Ft} is the total bequest that is observed in period t given by:

$$B_{Ft} = N_t(D)\left(\kappa^r_N b^r_{Ft} + \kappa^{om}_N b^{om}_{Ft} + \kappa^{oo}_N b^{oo}_{Ft} + \kappa^w_N b^w_{Ft}\right)$$

and where $b^j_{Ft} = \widehat{b}^j_F \mathrm{y}_{Lt}$ with $\widehat{b}^j_F$ defined in equation (86). Since $Y_{Nt} = (1-\alpha)Y_{Lt} = (1-\alpha)\mathrm{y}_{Lt}\int_A^R N_t(a)\,\mathrm{d}a$ one can conclude that:

$$b^N_{yFt} = \frac{B_{Ft}}{Y_{Nt}} = (1-\alpha)\widehat{b}_F\frac{N_t(D)}{\int_A^R N_t(a)\,\mathrm{d}a}$$

with $\widehat{b}_F = \left(\kappa_N^r \widehat{b}_F^r + \kappa_N^{om} \widehat{b}_F^{om} + \kappa_N^{oo} \widehat{b}_F^{oo} + \kappa_N^w \widehat{b}_F^w\right)$.

Total bequests include, however, not only the financial bequests B_{Ft} but also the bequest of the directly owned houses B_{Ht}, i.e. $B_t = B_{Ft} + B_{Ht}$. Under the assumption that the value of the directly owned housing stock is identical for all cohorts, the bequest of directly owned houses is simply given by the mortality rate times the value of the directly owned stock, i.e. $B_{Ht} = \mathfrak{m}_t P_{ht}^o \overline{H}_t^{od} = \mathfrak{m}_t \beta_{Hodt}^N Y_{Nt}$. where the mortality rate is constant and given by:

$$\mathfrak{m}_t = \frac{N_t(D)}{N_t} = \frac{e^{n(t-D)}}{e^{nt}\frac{e^{-nA}-e^{-nD}}{n}} = \frac{n}{e^{n(D-A)}-1} \equiv \mathfrak{m}^*. \tag{91}$$

For constant cohort sizes $\mathfrak{m}_t = \frac{1}{D-A}$. It thus follows that:

$$b_{Hyt}^N = \frac{B_{Ht}}{Y_{Nt}} = \beta_{Hodt}^N \mathfrak{m}_t = \beta_{Hodt}^N \mathfrak{m}^*$$

and:

$$b_{yt}^N = b_{Fyt}^N + b_{Hyt}^N = (1-\alpha)\widehat{b}_F \frac{N_t(D)}{\int_A^R N_t(a)\,\mathrm{d}a} + \beta_{Hodt}^N \mathfrak{m}^*.$$

The literature (Alvaredo et al. 2017) mostly reports the ratio of total bequest to national income (or to NDP) which is thus given by $b_{yt} = b_{yt}^N \frac{Y_{Nt}}{NDP_t}$.

B A simple analytical model to explain the empirical results

In this appendix I present a stylized model that can be helpful to understand the pattern of the numerical results of the paper and to emphasize the importance of the assumptions concerning the organization of the housing sector for the equilibrium portfolio distribution. In particular, I will show that and explain why the existence of outright owners can play a crucial role to match the increase in the share of housing wealth $\frac{\beta_H}{\beta}$ following a decline in interest rates.

In order to be able to do this in a tractable fashion I will shut down the demand side of the model (as presented in section 3 of the paper) and make two simplify assumptions. First, I assume that the equilibrium interest rate is given by $r = g+n+\omega$ where ω is some unspecified factor.[34] Second, it is assumed that both the renters and the buying owners

[34] In fact, this expression can be shown to be the outcome of a Kaldorian model in which normal

spend a constant share γ of their labor income on housing services, i.e. $P^r_{st}\overline{H}^r_t = \gamma Y^r_{Lt} = \gamma\kappa^r_N Y_{Lt}$ and $P^o_{st}\overline{H}^{om}_t = \gamma Y^o_{Lt} = \gamma\kappa^{om}_N Y_{Lt}$ (assuming identical incomes for both groups). Finally, I abstract here from the top 1% and focus on three groups of dwellers which correspond to the three segments of the housing stock: renters, owners with mortgages and direct owners with population shares κ^r_N, κ^{om}_N and κ^{oo}_N, respectively (with $\kappa^r_N + \kappa^{om}_N + \kappa^{oo}_N = 1$). The corresponding shares of the housing stock controlled by the three group of dwellers are denoted by κ^r_H, κ^{om}_H and κ^{oo}_H, respectively. From equations (14), (15) and (16) and $Y_{Lt} = (1-\alpha)Y_{Nt}$ it follows that the steady state housing-wealth-to-income ratios can be written as:

$$\beta^N_{Hr} = \frac{\gamma(1-\alpha)\kappa^r_N}{r_h + \delta_h - \widetilde{g}}, \beta^N_{Hom} = \frac{\gamma(1-\alpha)\kappa^{om}_N}{r_m + \delta_h - \widetilde{g}},$$

where $\beta^N_{Hom} = \frac{P^o_{ht}\overline{H}^{om}_t}{Y_{Nt}}$ is the value of the housing stock of the buying owners with mortgages. The house price formed in the segment of the self-buying owners is used to value stock of the directly owned houses. At the end of this appendix (in section B.1) I show that the ratio of the total owner-occupied housing stock comes out as: $\beta^N_{Ho} = \beta^N_{Hom} + \beta^N_{Hoo} = \frac{\gamma(1-\alpha)\kappa^{om}_N}{r_m+\delta_h-\widetilde{g}}\left(1 + \frac{\kappa^{oo}_H}{\kappa^{om}_H}\right)$. If it is furthermore assumed that the housing stocks of rental and self-bought houses adjust such that their service price is equal ($P^r_{st} = P^o_{st}$) one can calculate the ratio of housing wealth to capital wealth as:

$$\frac{\beta_H}{\beta_K} = \frac{\gamma(1-\alpha)}{\alpha}\left[\kappa^r_N\frac{r_k+\delta_k}{r_h+\delta_h-\widetilde{g}} + \left(1 - \kappa^r_N + \frac{\kappa^{oo}_H - \kappa^{oo}_N}{1-\kappa^{oo}_H}\right)\frac{r_k+\delta_k}{r_m+\delta_h-\widetilde{g}}\right]. \quad (92)$$

The housing share will thus depend on the relative size of the interest rates r_k, r_h and r_m, on the population share of renters κ^r_N and on the "relative housing abundance" $(\kappa^{oo}_H - \kappa^{oo}_N)$ of direct owners.

In general, it is hard to say how the portfolio ratio $\frac{\beta_H}{\beta_K}$ reacts to changes in the economic structure. For the highly stylized case with $r_k = r_h = r_m = r = g + n + \omega$, $\kappa^{oo}_H = \kappa^{oo}_N = 0$, $\widetilde{g} = g$ and $\delta_k = \delta_h = \omega = 0$ it holds that $\beta^N_K = \frac{\alpha}{g+n}$ and $\beta^N_H = \frac{\gamma(1-\alpha)}{n}$. A change in g will in this case only have an effect on β^N_K while a reduction in n will have an effect on both β^N_K and β^N_H where the latter effect will be larger effect. In particular it holds that:

$$\frac{\partial\left(\frac{\beta_H}{\beta_K}\right)}{\partial t} = \frac{\gamma(1-\alpha)}{\alpha}\frac{g}{n}\left(\frac{\dot{g}}{g} - \frac{\dot{n}}{n}\right).$$

renters and owners do not save and where only a tiny group of non-working wealth-holders is responsible for accumulation.

The effect of the change in the interest rate on the portfolio ratio thus depends on the source of the change. If the decline in r was primarily caused by a decrease in the growth rate g then the portfolio ratio will *decrease* while the opposite is true if a fall in population growth has been the main factor. For $\frac{\dot{g}}{g} = \frac{\dot{n}}{n}$ there is no change in the portfolio ratio.

For a realistic calibration with $\delta_k = 0.1$, $\delta_h = 0.025$ and $\omega = 0.05$ (such that the initial interest rate is $r = 9.5\%$) one would get a share that is equal to $\frac{\beta_H}{\beta} = 42\%$ for an initial situation with $g = 3\%$ and $n = 1.5\%$. This is reduced to $\frac{\beta_H}{\beta} = 41\%$ (after a reduction to $g = 2\%$), increased to $\frac{\beta_H}{\beta} = 44\%$ (after a reduction to $n = 0.5\%$) and basically kept equal at $\frac{\beta_H}{\beta} = 43\%$ after a reduction in both. The effect of changes in g and n on the share of housing wealth is thus rather small which is in line with the results of the numerical model. As shown more extensively in section B.1 below the effect is bigger for changes in the housing structure. In order to see this, assume a starting situation where the share of renters is 50%, there are no direct owners (i.e. $\kappa_N^r = \kappa_N^{om} = 50\%$ and $\kappa_N^{oo} = 0\%$) and where the spreads are given by $\xi_h = 0\%$ and $\xi_m = 2\%$. This situation is associated with a housing wealth share of $\frac{\beta_H}{\beta} = 45\%$. A decrease in the share of renters to 30% increases the share slightly to $\frac{\beta_H}{\beta} = 46\%$ while an increase in the spread to $\xi_m = 4\%$ causes a bigger increase to $\frac{\beta_H}{\beta} = 48\%$. On the other hand, starting from the same initial situation but now with $\kappa_N^{oo} = 20\%$ one can show that an increase in the share of directly owned houses from $\kappa_H^{oo} = 20\%$ to $\kappa_H^{oo} = 40\%$ increases $\frac{\beta_H}{\beta}$ from 45% to 53%. If the reduction in the renters' share, the increase in the spread and the change in the share of directly owned houses were to happen at the same time then $\frac{\beta_H}{\beta}$ would increase even further to 58%. The results of the simple model thus reflect the ones of calibrated model in sections 5.2 and 5.4 of the paper.

B.1 Details of the derivation of the simple example

This part of appendix B collects derivations for the simple example. Starting with the expression $\beta_{Hom}^N = \frac{\gamma(1-\alpha)\kappa_N^{om}}{r_m+\delta_h-\widetilde{g}}$ one can calculate the total housing wealth of owner-occupiers. To this end one can use the imputed rent $P_{st}^o = \frac{\gamma\kappa_N^{om}Y_{Lt}}{\overline{H}_t^{om}}$ which implies $P_{ht}^o = \frac{\gamma\kappa_N^{om}Y_{Lt}}{(r_m+\delta_h-\widetilde{g})\overline{H}_t^{om}}$. It thus follows that:

$$\begin{aligned}
\beta_{Ho}^N &= \frac{P_{ht}^o\overline{H}_t^o}{Y_{Nt}} = \frac{P_{ht}^o\left(\overline{H}_t^{om}+\overline{H}_t^{oo}\right)}{Y_{Nt}} = \frac{\gamma(1-\alpha)\kappa_N^{om}}{(r_m+\delta_h-\widetilde{g})\overline{H}_t^{om}}\left(\overline{H}_t^{om}+\overline{H}_t^{oo}\right) \\
&= \frac{\gamma(1-\alpha)\kappa_N^{om}}{r_m+\delta_h-\widetilde{g}}\left(1+\frac{\kappa_H^{oo}}{\kappa_H^{om}}\right)
\end{aligned}$$

where the last line uses the definition in (3) and where (due to the absence of the top 1% I now have $\overline{H}_t^{od} = \overline{H}_t^{oo}$ and $\kappa_H^{od} = \kappa_H^{oo}$). In order for renters and buying owners to be indifferent between their choice of dwelling it is required that in the long-run the shares of the housing stocks κ_H^r and κ_H^{om} adjust such that the rent of the two housing forms is identical, i.e. such that $P_{st}^r = P_{st}^o$. Using the equations above this amounts to $\frac{\kappa_N^r}{\kappa_H^r} = \frac{\kappa_N^{om}}{\kappa_H^{om}}$. In fact, this is the same as equation (38) under the assumption that the utility of renting and owning is identical ($\eta^o = \eta^r$). For a given population share of outright owners κ_N^{oo} and a given share of outrightly owned houses κ_H^{oo} one can derive the equilibrium shares of rented and self-acquired houses as:

$$\kappa_H^r = \kappa_N^r \frac{1-\kappa_H^{oo}}{1-\kappa_N^{oo}}, \kappa_H^{om} = \kappa_N^{om} \frac{1-\kappa_H^{oo}}{1-\kappa_N^{oo}} \tag{93}$$

which corresponds to equation (40) for $\eta^o = \eta^r$. If there are no direct owners ($\kappa_H^{oo} = \kappa_N^{oo} = 0$) or if the share of directly owned houses also corresponds to their population share ($\kappa_H^{oo} = \kappa_N^{oo}$) then the service price for rented and self-acquired houses is the same if $\kappa_H^r = \kappa_N^r$ and $\kappa_H^{om} = \kappa_N^{om}$. Using (93) one can calculate that:

$$\begin{aligned}
\beta_{Ho}^N &= \frac{\gamma(1-\alpha)\kappa_N^{om}}{r_m+\delta_h-\widetilde{g}}\left(1+\frac{\kappa_H^{oo}}{\kappa_H^{om}}\right) = \frac{\gamma(1-\alpha)\kappa_N^{om}}{r_m+\delta_h-\widetilde{g}}\left(1+\frac{\kappa_H^{oo}}{\kappa_N^{om}}\frac{1-\kappa_N^{oo}}{1-\kappa_H^{oo}}\right) \\
&= \frac{\gamma(1-\alpha)}{r_m+\delta_h-\widetilde{g}}\left(\kappa_N^{om}+\kappa_H^{oo}\frac{1-\kappa_N^{oo}}{1-\kappa_H^{oo}}\right) = \frac{\gamma(1-\alpha)}{r_m+\delta_h-\widetilde{g}}\left(1-\kappa_N^r+\frac{\kappa_H^{oo}-\kappa_N^{oo}}{1-\kappa_H^{oo}}\right)
\end{aligned}$$

where the last line uses $\kappa_N^{om} = 1-\kappa_N^r-\kappa_N^{oo}$. The total housing wealth ratio $\beta_H^N = \beta_{Hr}^N+\beta_{Ho}^N$ can thus be calculated as:

$$\beta_H^N = \frac{\gamma(1-\alpha)\kappa_N^r}{r_h+\delta_h-\widetilde{g}} + \frac{\gamma(1-\alpha)}{r_m+\delta_h-\widetilde{g}}\left(1-\kappa_N^r+\frac{\kappa_H^{oo}-\kappa_N^{oo}}{1-\kappa_H^{oo}}\right)$$

and the ratio of housing wealth to physical wealth as equation (92) which is here repeated:

$$\frac{\beta_H}{\beta_K} = \frac{\gamma(1-\alpha)}{\alpha}\left[\kappa_N^r\frac{r_k+\delta_k}{r_h+\delta_h-\widetilde{g}} + \left(1-\kappa_N^r+\frac{\kappa_H^{oo}-\kappa_N^{oo}}{1-\kappa_H^{oo}}\right)\frac{r_k+\delta_k}{r_m+\delta_h-\widetilde{g}}\right]. \tag{92}$$

As noted above, the housing share depends on a number of parameters and in general it is not clear how it will react to changes in the economic structure. In order to delve deeper into this issue it is therefore instructive to start with a very simple example. In particular, assume that all interest rates are equal and given by $r_k = r_h = r_m = r = g+n+\omega$. In addition assume that there are no direct owners ($\kappa_H^{oo} = \kappa_N^{oo} = 0$), that $\widetilde{g} = g$ and that

$\delta_k = \delta_h = \omega = 0$. As reported above, it then holds that:

$$\frac{\partial \left(\frac{\beta_H^N}{\beta_K^N} \right)}{\partial t} = \frac{\gamma(1-\alpha)}{\alpha} \frac{g}{n} \left(\frac{\dot{g}}{g} - \frac{\dot{n}}{n} \right).$$

The effect of the change in the interest rates on the portfolio ratio thus depends on the source of the change. If the decline in r was primarily caused by a decrease in the growth rate g then the portfolio ratio will *decrease* while the opposite is true if a fall in population growth has been the main factor. For $\frac{\dot{g}}{g} = \frac{\dot{n}}{n}$ there is no change in the portfolio share. To get a feeling for the quantitative dimensions involved, assume $\alpha = 1/3$, $\gamma = 0.17$, $g = 3\%$ and $n = 1.5\%$. This implies a portfolio ratio of $\frac{\beta_H}{\beta_K} = 102\%$ and a portfolio share of $\frac{\beta_H}{\beta} = \frac{\beta_H}{\beta_K + \beta_H} = 51\%$. A fall in the productivity growth rate to $g = 2\%$ reduces the share to $\frac{\beta_H}{\beta} = 44\%$ while a decline of population growth to $n = 0.5\%$ leads to an increase to $\frac{\beta_H}{\beta} = 70\%$. The considerable size of these effects is, however, due to the specific assumptions. If one assumes, e.g., that $\delta_k = 0.1$, $\delta_h = 0.025$ and $\omega = 0.05$ (such that the initial interest rate is $r = 9.5\%$) then one would get an initial share that is equal to $\frac{\beta_H}{\beta} = 42\%$ which is reduced to $\frac{\beta_H}{\beta} = 41\%$ (after a reduction in g), increased to $\frac{\beta_H}{\beta} = 44\%$ (after a reduction in n) and basically unchanged after after reductions in both. This result is a mirror image of the results of the calibrated model for which the observed changes in the main economic parameters were typically associated with small and often almost no changes in the share of housing wealth.

It is, however, possible to get more sizable changes in the housing shares if one re-introduces a more differentiated picture of the housing market as argued in the following. In order to see this I assume an initial situation where the share of renters is 50%, there are no direct owners (i.e. $\kappa_N^r = \kappa_N^{om} = 50\%$ and $\kappa_N^{oo} = 0\%$) and where the spreads are given by $\xi_h = 0\%$ and $\xi_m = 2\%$. This situation is associated with a housing wealth share of $\frac{\beta_H}{\beta} = 45\%$.[35]

Decrease in the share of renters: If the share of renters κ_N^r decreases from 50% to 30% this is associated with a slight increase in the housing share from $\frac{\beta_H}{\beta} = 45\%$ to $\frac{\beta_H}{\beta} = 46\%$. Note that a change in the share of renters and self-buying owners has only an effect in the share if $\xi_h \neq \xi_m$ (or $r_h \neq r_m$), otherwise equation (92) reduces to $\frac{\beta_H}{\beta_K} = \frac{\gamma(1-\alpha)}{\alpha} \frac{r_k + \delta_k}{r_h + \delta_h - \widetilde{g}}$.

[35]This has to be evaluated numerically due to the fact that I assume $r = g + n + \omega$ which is itself a weighted average of the various interest rates, i.e. $r = \frac{r_k \beta_K + r_h \beta_{Hr} + r_m \beta_{Hom}}{\beta_K + \beta_{Hr} + \beta_{Hom}}$.

Increase in the spread: As a second case one can consider the situation where the share of renters and self-buyers is again 50% but where the risk discount of the mortgage interest rate increases from $\xi_m = 2\%$ to $\xi_m = 4\%$. This change would increases the share by 2.9 percentage points to $\frac{\beta_H}{\beta} = 48\%$.

Existence of direct owners: In order to discuss the impact of direct owners it is best to start from the situation where $r_m = r_h$. In this case one can use equation (92) to simplify that:

$$\frac{\beta_H}{\beta_K} = \frac{\gamma(1-\alpha)}{\alpha} \frac{r_k + \delta_k}{r_h + \delta_h - \widetilde{g}} \left[1 + \frac{\kappa_H^{oo} - \kappa_N^{oo}}{1 - \kappa_H^{oo}}\right] = \frac{\gamma(1-\alpha)}{\alpha} \frac{r_k + \delta_k}{r_h + \delta_h - \widetilde{g}} \left[\frac{1 - \kappa_N^{oo}}{1 - \kappa_H^{oo}}\right]. \quad (94)$$

In the case where the share of the directly owned housing stock is identical to the population share of the direct owners then this has no effect on the equilibrium prices and equilibrium housing portfolio share. The existence of the directly owned segment has no effect on the equilibrium allocation since the same number of houses and house owners is removed from the market. If, however, the direct owners possess houses that are on average larger than the average houses in the rest of the market (i.e. if $\kappa_H^{oo} > \kappa_N^{oo}$) then this has an effect on prices and the housing share. If, in the extreme case, the direct owners command over almost the entire housing stock $\kappa_H^{oo} \to 1$ the share $\frac{\beta_H}{\beta_K}$ goes to infinity. The eager buyers drive the price P_{st}^o to astronomic heights, thereby also increasing the value of non-traded housing stock of the direct owners. As an example assume that $\xi_h = \xi_m = 2\%$ where in the absence of direct owners one had that $\frac{\beta_H}{\beta} = 47\%$. If the share of direct owners is increased to $\kappa_N^{oo} = 20\%$ then the effect depends on the size of the associated housing stock. If $\kappa_H^{oo} = 20\%$ then there is no effect and still $\frac{\beta_H}{\beta} = 47\%$. If, however, $\kappa_H^{oo} = 40\%$ then the housing share is considerably higher at $\frac{\beta_H}{\beta} = 54\%$.

The quantitative effect is similar if one returns to the previous example with $\xi_h = 0\%$, $\xi_m = 2\%$. For the case with $\kappa_N^{oo} = 20\%$ an increase in the housing share of direct owners from $\kappa_H^{oo} = 20\%$ to $\kappa_H^{oo} = 40\%$ now increases the portfolio share from $\frac{\beta_H}{\beta} = 45\%$ to $\frac{\beta_H}{\beta} = 53\%$ (for $\kappa_N^r = 50\%$).[36]

Decrease in the share of renters and increase in the spread and a change in the share of directly owned houses: Looking at a situation where all three changes

[36]Note that in the case with $\xi_m \neq \xi_h$ there is now a tiny difference even for $\kappa_H^{oo} = \kappa_N^{oo}$. This is due to the fact that now the share κ_N^{om} is slightly lower and this changes somewhat the value of r_k and thus also of r_h and r_m.

happen at the same time (a reduction from $\kappa_N^r = 50\%$ to $\kappa_N^r = 30\%$, an increase from $\xi_m = 2\%$ to $\xi_m = 4\%$ and a change from $\kappa_H^{oo} = 20\%$ to $\kappa_H^{oo} = 40\%$ with $\kappa_N^{oo} = 20\%$) now leads to an increase to $\frac{\beta_H}{\beta} = 57.8\%$.

It can be shown that also the effect of changes in g and n is amplified if there is a situation with $\xi_m > \xi_h$ and $\kappa_H^{oo} > \kappa_N^{oo}$.

C Calibration

C.1 Main parameters

The steady-state comparison in section 5 is based on data from the Worldbank and from the OECD. The values for g (real GDP growth) and n (population growth) correspond to the data for the group of high income countries in the World Development Indicators database and are the geometric average for the periods 1976-1984 (for the initial situation) and 2015-2021 (for the current situation). The data for A, R and D are based on the assumptions used in Summers & Rachel (2019) while the values for ρ follow OECD data.

As far as housing is concerned, one can start with the share of housing-related expenditure as a percentage of total household consumption expenditures in the OECD Affordable Housing Database (Figure HC 1.1.3) which is reported as 22.8% for the OECD-average. This number, however, also includes expenditures on electricity, gas, water etc. If one only considers the numbers for actual and imputed rents the expenditure share comes out as 16.7% or (if one also adds the expenditures for maintenance and repair of the dwelling) as 17.5%. These data thus suggest the choice of $\gamma = 0.17$.

The depreciation rate of housing structures is often assumed to be 1.5% (Kaplan et al. 2020, Sommer & Sullivan 2018, Grossmann, Larin, Löfflad & Steger 2021). A number of papers, however, also include housing-related taxes which are assumed to be around 1% (Kaplan et al. 2020, Sommer & Sullivan 2018) and I thus use $\delta_h = 2.5\%$.

The depreciation rate for physical capital is in line with the value of 6.8% (an average from 1970 to 2019) used in McKay & Wieland (2021) for durable assets which includes in their definition also residential housing. A value around $\delta_k = 10\%$ is thus in line with a share of housing wealth between 40% and 50% (in particular: $6.8\% = 0.42 \times 2.5\% + 0.58 \times 10\%$).

The calibration for the inequality of income ($d_y^w = 3$) follows from Table C.7 in the appendix to Garbinti et al. (2020) where the authors report for France a labor income share of around 3% for the top 1% wealth group in the 1980ies. In the same source the total income shares (including capital income) are reported to be around 8% for the top 1% and around 26% for the top 10%. In order to get a cross-country picture one can look at the data for the US reported in Kuhn et al. (2020, Table E.4) which, however, only refer to the total income share of the top 10%. Since the number (26.7%) is close to the one from French I use the corresponding figure for the labor income share for the top 1% for the calibration. For the later period I use a value of $d_y^w = 4.5$. In fact, Garbinti et al. (2020) report that the share of *labor* income of the top 1% in France seemed to have declined between 1970 and 2014 and at most stayed constant at 3% since 1990 which would suggest to leave the parameter value at $d_y^w = 3$. On the other hand, however, the data show group-specific differences in the returns to capital which are absent from my model. The increase in d_y^w is also meant to capture this trend. Also the data from Kuhn et al. (2020) for the US show an increase in the share of total income for the top 10% by about 50%. I want to note, however, that one could alternatively keep the value at value at $d_y^w = 3$ and simply assume a stronger increase in the bequest motive in order to capture the increase in the wealth share of the top 1% to 35%. This leads to basically identical results for r, β and β_h/β. The only difference is in the share of inherited wealth.

For the risk discounts I rely on the data in Jordà et al. (2019). First, one has to note that the existing data on the rate of return on housing investments do not support the assumption of large discounts ξ_h. Jordà et al. (2019), e.g., show that the rates of returns on equity and on housing are very similar over a longer time period. In fact, taken the lower volatility of house prices into account Jordà et al. (2019) argue that the risk-adjusted returns of housing are even larger than the ones of equity investment. For mortgage rates, however, it is more reasonable and in line with the empirical evidence to assume considerable risk discounts. Im particular, the available evidence suggests that mortgage interest rates are typically between 2 pp to 3 pp above government bonds rates which itself are assumed to have a risk discount of around 5 pp. I thus assume in the initial calibration that $\xi_d = 5\%$ and $\xi_m = 2\%$ and assume for the calibration of the later period that the mortgage discount rate increases to $\xi_m = 3\%$.

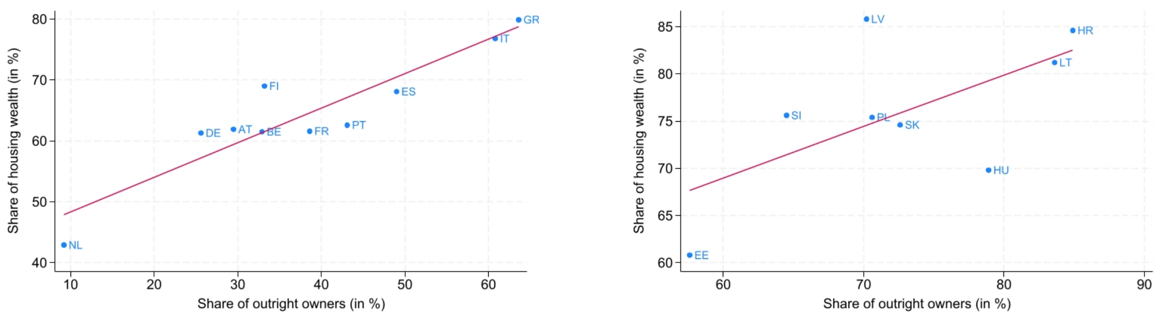
For the utility gain for owning η^o I follow Iacoviello & Pavan (2013). They use a similar model and calibrate a utility penalty for renting of 0.838 in order to obtain a homeownership rate of 64% as in the data for the period 1952–1982 in the US (p.227). I therefore choose a value of $\eta^0 = 1.2 \approx 1/0.838$.

C.2 Cross-country evidence on wealth and outright owners

The four group model of section 5.2 distinguishes the bottom 99% into three group of dwellers: renters, owner-occupiers with mortgages and outright owners. In order to calibrate the model and to compare the implied steady state around 1980 with the steady state around 2018 it would be best to use data on the composition of dwellers and the characteristics of their dwellings both over time and across countries. Unfortunately, these data do not seem to be available and it is therefore necessary to piece together a number of data source in order to approach this task.

Data from the English Housing Survey, e.g., report that the share of households who rent increased from 16% in 1981 to around 23% in 2018. At the same time the share of owners with a mortgage decreased from 47% to 35% while the share of outright owners increased from 37% to 41%. The data from the SFC+ (Kuhn et al. 2020) show a slightly different picture for the US where the share of renting households stayed more or less constant at 36% between 1983 and 2016 while the share of owners with mortgages increased from 37% to 42% and the share of outright owners decreased from 26% to 22%. OECD data are only available for the time span from 2010 to 2020 and they show only little movements with the (unweighted) average share of renters staying at 18%-19% and the average shares of owners with and without a mortgage at 23% and 48%, respectively. All of these data refer, however, only to the population share of dwellers (to κ_N^j in my notation) and not to the relative value of the housing stock that they control (to κ_H^j in my notation).

There exists, however, indirect evidence that the housing structure is important for the wealth-to-income ratios and the share of housing wealth. In particular, in a standard model the tenure choice would not have an impact on the accumulation or composition of wealth (since renters and owners would, e.g., adjust their financial wealth such as to meet an identical target of total wealth). This, however, is not reflected in the available data. In Figure C1 I plot the share of outright owners vs. the share of housing wealth for a group of Western and Eastern European countries. The housing wealth variable comes from the fourth wave of the HFCS and is defined as the sum of the value of the household's main residence plus the value of other real estate minus the value of outstanding mortgage debt divided by the net wealth. Using gross values in both the numerator and the denominator give very similar values for the housing share. The share of outright owners come from Table HM1.3.3 in the OECD's Affordable Housing Database. For both groups of countries an increase in the share of outright owners by 10 pp increases the share of housing wealth



(a) Western European (b) Eastern European

Figure C1: The data come from the HFCS (4th wave, 2017) and OECDs (Affordable Housing Database, 2020 or latest year available).

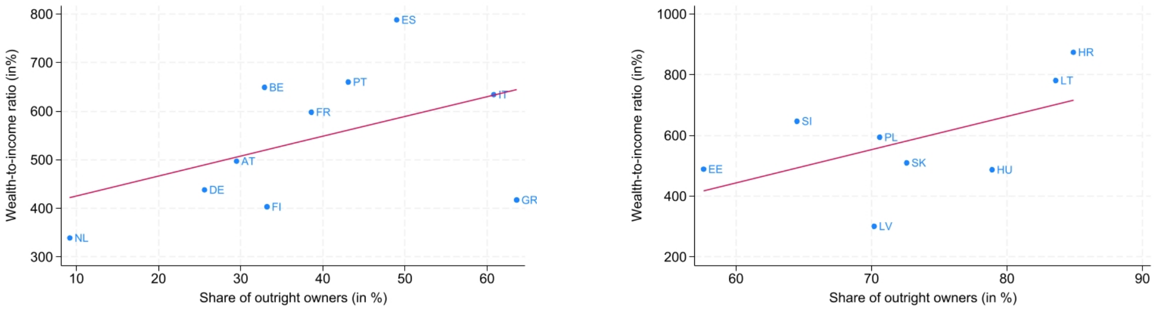
by about 5 pp (in other words, the slope of the regression line is around 0.5). A similar picture emerges if one contrasts the share of outright owners with the total wealth-to-income ratio. Also here one gets a positive correlation although now the relation is less clear-cut and the regression coefficients are no longer statistically significant.

The correlation between the share of outright owners and the share of housing wealth in Figure C1 can be used to get an idea about reasonable values for the calibration. In particular, one can use the formula (92) (or the corresponding expression for $\frac{\beta_h}{\beta}$) for the simple example in appendix B to investigate which value of $\frac{\kappa_H^{od}}{\kappa_N^{od}}$ (where $\kappa_N^{od} = \kappa_N^{oo} + \kappa_N^{w}$) gives a relation that is in line with the empirical evidence. In the theoretical model the relation is non-linear but it appears that values between $\frac{\kappa_H^{od}}{\kappa_N^{od}} = 1.5$ and $\frac{\kappa_H^{od}}{\kappa_N^{od}} = 1.75$ give reasonable results (i.e. an increase in the share of outright owners by 10 pp increases the share of housing wealth by around 5 pp). This is the rationale behind the chosen calibration of 1.65 in section 5.2 of the paper. Repeating this exercise for the entire model leads to similar results. A thorough investigation of these empirical issues is a topic for future research.

D Numerical examples

D.1 Alternative specifications

In sections 5.2 to 5.3 I have argued why it is necessary to use the full four group model (and the assumption of a changing influence of outright owners) in order to explain the level and path of a number of crucial macroeconomic variables, starting with the share of



(a) Western European (b) Eastern European

Figure C2: The data come from the HFCS (4th wave, 2017) and OECDs (Affordable Housing Database, 2020 or latest year available).

housing wealth. In this appendix I investigate whether and in how far other parameter changes could be invoked in order to get similar results. In Tables D1 and D2 I start with the same benchmark calibration as in Table 1 but I leave the share of dwellers and their controlled housing stock constant while studying the impact of alternative parameter calibrations concerning the non-housing sector, the housing sector and preferences. In Table D3 I then come back to the role of the composition of dwellers and I deal with different assumptions concerning the role of outright owners.

Non-housing sector: In Table D1 I investigate whether one could invoke changes in the structure of the production side or the housing market in order to explain the increase in the share of housing wealth over the last decades. To this end I shut down the channels used in section 5.2 to explain this phenomenon (see line 2 in Table 1 here repeated as line 2a), i.e. I use a reference case where the population shares κ_N^j and the associated shares of houses κ_H^j stay constant. By comparing lines 2a and 2b of Table D1 one can see that the effect on the interest rate is rather similar ($r = 5.63\%$ vs. $r = 5.69\%$) while the impact on the wealth-to-income ratio shows a larger difference ($\beta = 521\%$ vs. $\beta = 599\%$). The most significant difference, however, is that in the benchmark scenario (involving changes in the importance of outright owners) the share of housing wealth increases to $\frac{\beta_H}{\beta} = 54\%$ while in the current case it decreases to $\frac{\beta_H}{\beta} = 44\%$. In the lower part of Table D1 I focus on the question whether alternative assumptions about production or the housing market could be used to explain the increasing share of housing wealth.

The results of lines 3 and 4 in Table D1 might give the impression that a change in the production structure could contribute to an understanding of the observed facts. In

Table D1: Alternative specifications 1

Nr.	Case	r	β	$\frac{\beta_H}{\beta}$	$\overline{s}$	$\overline{s}^{net}$	$\overline{s}^{NA}$	$\overline{s}^{NA,net}$	b_{Fy}	b_y
1	**Initial**	**9.64%**	**350%**	**46%**	**29.9%**	**15.3%**	**28.2%**	**12.9%**	**5.9%**	**6.6%**
2a	**Today**	**5.69%**	**599%**	**54%**	**33.9%**	**15.1%**	**31.4%**	**11.%**	**7.1%**	**9.2%**
2b	**Today**	**5.63%**	**521%**	**44%**	**30.9%**	**12.%**	**29.6%**	**9.8%**	**7.1%**	**8.1%**
		Changes in the non-housing sector								
3	$\alpha = 0.25$	5.14%	501%	55%	27.3%	11.1%	25.4%	8.2%	5.4%	6.7%
4	$\delta_k = 10\%$	5.36%	513%	51%	33.%	11.3%	31.5%	8.6%	6.1%	7.3%
		Changes in the housing sector								
5	$\gamma = 0.22$	5.87%	573%	52%	32.2%	13.4%	30.3%	10.4%	8.1%	9.4%
6	$\chi = 0.1$	5.73%	548%	48%	31.4%	12.5%	29.3%	9.%	7.5%	8.7%
7	$\delta_h = 0\%$	6.06%	623%	58%	29.1%	15.7%	26.3%	11.7%	8.9%	10.7%
8	$\xi_h \uparrow$, $\xi_m \uparrow$	5.79%	560%	51%	31.%	12.2%	29.2%	9.4%	7.8%	9.%

Note: The table shows various alternative specifications for a four group model that corresponds in the initial situation to the one of Table 1 with $\kappa_N^r = 50\%$, $\kappa_N^{om} = 25\%$, $\kappa_N^{oo} = 24\%$, $\kappa_N^w = 1\%$ and where $\kappa_H^{oo} = \kappa_N^{oo}$, $\kappa_H^w = \kappa_N^w$. Different to Table 1 it is, however, assumed that the values for κ_N^j and κ_H^j stay constant between the initial and current situation. The results for the current situation are shown in line 2b (while the one for the benchmark case from Table 1 are reported in line 2a). The results in lines 3 to 8 always refer to the current situation where all parameters are assumed to take on the same values as in line 2b with an additional parameter change as indicated. In line 8 it is assumed that $\xi_h = 2.5\%$, $\xi_m = 4.5\%$.

particular, a decrease in the weight of physical capital in the production function (from $\alpha = 1/3$ to $\alpha = 1/4$, probably due to a shift in technology) or an increase in the capital depreciation rate (from $\delta_k = 10\%$ to $\delta_k = 15\%$, probably due to faster obsolescence of capital goods) implies a further reduction in the interest rate by an additional 0.5 percentage points and also a sizeable increase in the share of housing in the wealth portfolio (from 46% to 55% or 51%). At the same time, however, one has to note that the change in the wealth-to-income ratio is now more modest and not completely in line with the observed facts. Furthermore, it is hard to argue that such fundamental changes in the production structure as suggested by the assumed parameter changes could in fact be observed for the advanced countries over the last decades.

Housing sector: One could argue that the preference for housing has increased over the last decades. As shown in line 5 of Table D1 a larger importance of housing ($\gamma = 0.22$) increases the value of the housing stock and thereby the housing portfolio share to 52%. The existing data on the share of housing expenditures are, however, not in line with this assumed increase in γ. Furthermore, this assumption leads to a reduction in capital investments which *dampens* the reduction in the interest rate. Another interesting case is the assumption of a decline in the parameter χ, i.e. in the extent with which the housing supply reacts to population growth. A decline in χ could e.g. be the result of sluggish housing construction, arguably due to overly strict zoning laws or to NIMBY attitudes. As shown in line 6 of Table D1, however, the equilibrium for $\chi = 0.1$ is not much affected by this parameter change.

The expression for the house prices $P^r_{ht} = \frac{P^r_{st}}{r_h+\delta_h-\widetilde{g}}$ and $P^o_{ht} = \frac{P^o_{st}}{r_m+\delta_h-\widetilde{g}}$ (see (8)) indicates two more candidates that could lead to a higher equilibrium valuation of houses and thus to a higher share of housing wealth. First, as shown in line 7 of Table D1, a reduction in δ_h (which could, e.g., capture a higher attractiveness of housing via the tax system) leads to a higher share of housing wealth (58%). At the same time, however, this change also dampens the reduction in the interest rate. This is due to the fact that the lower value of δ_h makes housing a more attractive investment thereby crowding out physical capital. Equivalently, one could also increase the risk discounts by the same amount (2.5%) as the reduction in the depreciation rate. Since in the initial situation in line 1 of Table D1 it is assumed that the risk discount only applies to mortgages ($\xi_m = 2\%$ while $x_h = 0\%$) I assume for line 8 that $\xi_m = 4.5\%$ and $x_h = 2.5\%$. This has a qualitatively similar effect on the main variables as the reduction in δ_h although now the impact on the interest rate is larger (5.8% vs. 6.1%) while the one on the share of housing wealth share is weaker

(51% vs. 58%).

In order to get a larger reaction of the main variables one could of course assume even larger discounts ξ_h and ξ_m which would increase the housing share. For $\xi_h = 5\%$ and $\xi_m = 7\%$ one gets for example a housing share of 58% (together with a wealth-ratio of $\beta = 608\%$ and an interest rate of $r = 6.0\%$). The problem with the assumption of such high risk discounts is, however, that they are not in line with the empirical evidence, in particular in as far as ξ_h is concerned. Jordà et al. (2019), e.g., have shown that the rates of returns on equity and on housing are very similar. In fact, taken the lower volatility of house prices into account Jordà et al. (2019) have argued that the risk-adjusted returns of housing are even larger than the ones of equity investment. Later studies based on more detailed data (Chambers et al. 2021, Eichholtz et al. 2021) have challenged some of these findings (especially the one about the superiority of risk-adjusted returns of housing), but even these studies do not seem to support risk discounts on residential housing investments that are larger than 2-3%.

Table D2: Alternative specifications 2

Nr.	**Case**	**r**	$\boldsymbol{\beta}$	$\boldsymbol{\frac{\beta_H}{\beta}}$	$\overline{s}$	$\overline{s}^{net}$	$\overline{s}^{NA}$	$\overline{s}^{NA,net}$	$\boldsymbol{b_{Fy}}$	$\boldsymbol{b_y}$
1	**Initial**	**9.64%**	**350%**	**46%**	**29.9%**	**15.3%**	**28.2%**	**12.9%**	**5.9%**	**6.6%**
2	**Today**	**5.69%**	**599%**	**54%**	**33.9%**	**15.1%**	**31.4%**	**11.%**	**7.1%**	**9.2%**
		$\theta = 0.02$, $\sigma = 2$								
3	Initial	8.76%	381%	48%	31.5%	16.3%	29.7%	13.6%	3.8%	4.6%
4	Today	5.55%	613%	54%	34.3%	15.3%	31.7%	11.%	6.4%	8.6%
		$\theta = -0.0025$, $\sigma = 1.5$								
5	Initial	6.26%	499%	54%	37.2%	19.9%	34.7%	15.8%	1.4%	2.5%
6	Today	3.84%	826%	62%	39.5%	17.4%	35.9%	10.6%	2.6%	6.%
		$\theta = -0.0025$, $\sigma = 1.5$, $s_B^r = 0.09$								
7	Initial	4.58%	645%	60%	43.%	23.9%	39.6%	17.8%	6.6%	8.3%
8	Today	2.66%	1100%	69%	44.5%	19.3%	39.7%	8.7%	8.2%	13.6%
		$g_\varepsilon = g$, $s_B^r = 0.09$								
9	Initial	4.96%	603%	58%	41.4%	22.8%	38.3%	17.2%	7.5%	9.1%
10	Today	2.89%	1030%	67%	43.4%	18.9%	38.8%	9.3%	8.9%	13.8%

Note: The table shows various alternative assumptions related to the preference parameters. Rows 1 and 2 contain the benchmark model with $g_\varepsilon = g$ and $s_B = 0.0089$, $s_B^w = 0.093$ (initial) and $s_B^w = 0.207$ (current). The other assumptions are indicated in the table.

Preferences: For the benchmark calibration I have assumed that the parameters describing the time preference θ and σ adjust such that $g_\varepsilon = g$. As noted by Piketty (2010, p.138) there exist good reasons (e.g. uncertainty about future growth and the existence of borrowing constraints) why in the real world expenditure growth is often relatively close to g. In the related literature, however, the parameters θ and σ are mostly set equal to standard values or chosen such as to meet target values for r or β. Summers & Rachel (2019), e.g., set $\theta = 0.02$ and $\sigma = 2$ while Platzer & Peruffo (2022) choose $\sigma = 1.5$ and $\theta = -0.0025$ (a *negative* rate of time preference) in order to target a low real interest rate of $r = 0.53\%$.

In lines 3 and 4 of Table D2 I show the outcome of a standard calibration with $\theta = 0.02$ and $\sigma = 2$ again both for the initial and the current situation. The results are similar to the outcome of the benchmark scenario in Table 1 (here repeated as lines 1 and 2) based on $g_\varepsilon = g$ with only a somewhat lower interest rate (8.8% vs. 9.6%) and an associated somewhat higher wealth-to-income ratio. This has to do with the fact that for $\theta = 0.02$ and $\sigma = 2$ the expenditure growth rate comes out as $g_\varepsilon = 3.7\%$ which is larger than the income growth rate of $g = 3\%$. In order to implement this steeper expenditure growth path households will thus undertake some extra saving which lowers r and increases β. The effect is qualitatively similar but much larger in size for the more extreme case with $\theta = -0.0025$ and $\sigma = 1.5$ as shown in lines 5 and 6. Now the interest rate in the initial situation is only $r = 6.3\%$ and the wealth-to-income ratio is also increased to $\beta = 499\%$. This, however, is associated with a very high expenditure growth rate of $g_\varepsilon = 4.7\%$ and high implied savings rates of 37.2% (gross) and 19.9% (net). For the "current situation" the calibration implies a value of $r = 3.84\%$ which implies a high wealth-to-income ratio of $\beta = 826\%$.[37]

The bequest ratio for the assumption $\theta = -0.0025$ and $\sigma = 1.5$ with the benchmark calibration of the strength of the bequest motive is counterfactually low ($< 3\%$). As shown in lines 7 and 8 of Table D2 the additional assumption of $s_B = 0.09$ leads to a higher bequest ratio together with a further reduction in the interest rate (to $r = 2.66\%$). The wealth-to-income ratio and the savings rates for this calibration look, however, excessively

[37] The reason why my model does not result in the same equilibrium interest rate of $r = 0.54\%$ has to do with various additional differences between the models. Platzer & Peruffo (2022), e.g., assume non-homothetic utility functions, a different formulation for the bequest motive, a mark-up in the production process and—above all—they abstract from a housing sector which in my case also works against an "overaccumulation" of physical capital. As shown in section 5.3 for the assumption of $\gamma = 0$ (no housing) the calibration that targets a level of $\beta = 350\%$ in the initial situation is associated with comparably low (or even lower) interest rates of $r = 3.3\%$ (initial) and $r = -1.32\%$ (current) in my set-up.

large. For the sake of comparisons, lines 9 and 10 document the impact of an increase in s_B for the benchmark case with $g_\varepsilon = g$ which also leads to rather implausible values for β and $\overline{s}$.

Different assumptions about outright owners: As noted in section 5.1 of the paper and in appendix C, the available data on the share of the housing stock that is in the possession of the outright owners is scarce. For the benchmark calibration in Table 1 I used the simple assumption that half of the households are owners, half of the owners are outright owners and that in the initial situation this also corresponds to their share of the housing stock while this percentage is increased by 65% when compared to the current situation. Expressed formally this means: $\kappa_N^r = 50\%$, $\kappa_N^{om} = 25\%$, $\kappa_N^{oo} = 24\%$, $\kappa_N^w = 1\%$ and $\kappa_H^{oo} = 24\%$, $\kappa_H^w = 1\%$ (initial), $\kappa_H^{oo} = 48\%$, $\kappa_H^w = 1.65\%$ (today).

In Table D3 I look at three alternative assumptions about the development of the housing stock controlled by the outright owners. In the first alternative specification I assume that this housing stock increases by 100% (i.e. in the current situation $\kappa_H^{oo} = 48\%$, $\kappa_H^w = 2\%$). In the second scenario, I assume that already in the initial situation the outright owners control a overproportional share of the housing stock ($\kappa_H^{oo} = 28.8\%$, $\kappa_H^w = 1.2\%$) that further increases in the current situation ($\kappa_H^{oo} = 42\%$, $\kappa_H^w = 1.75\%$). In the third scenario, I finally assume that the fraction ($\kappa_H^{oo}/\kappa_N^{oo} = \kappa_H^w/\kappa_N^w = 120\%$ stays constant over time but that the share of outright owners in the population increases from $\kappa_H^{oo} = 24\%$ to $\kappa_N^{oo} = 34\%$. In all cases the bequest motive of the top 1% is adjusted such that the share of top 1% wealth continues to move from 28% to 35%.

The assumption of a doubling of the housing stock controlled by outright owners (line 4) implies an even larger decrease in the interest rate to 4.8% and an even larger increase in the wealth-to-income ratio and the share of housing wealth (to 61%). These values as well as the savings rates seem, however, too large to depict a plausible scenario. The second alternative scenario, on the other hand, where the outright-owner-controlled share moves from 30% to 44% (while the move is from 25% to 41% in the baseline scenario) shows similar results as the benchmark case in line 2. Under the third assumption the decrease in the interest rate and the increase in the wealth-to-income ratio are larger than in the benchmark scenario. In this case, however, the share of housing wealth shows only a smaller increase from 48% to 52%. Altogether these results underline the crucial role played by the assumption about the outright owners and their housing stock.

Table D3: Alternative specifications 3

Nr.	Case	r	β	$\frac{\beta_H}{\beta}$	$\overline{s}$	$\overline{s}^{net}$	$\overline{s}^{NA}$	$\overline{s}^{NA,net}$	b_{Fy}	b_y
		Benchmark model (Today: $\kappa_H^{oo} = 48\%$, $\kappa_H^{w} = 2\%$)								
1	**Initial**	**9.64%**	**350%**	**46%**	**29.9%**	**15.3%**	**28.2%**	**12.9%**	**5.9%**	**6.6%**
2	**Today**	**5.69%**	**599%**	**54%**	**33.9%**	**15.1%**	**31.4%**	**11.%**	**7.1%**	**9.2%**
		Today: $\kappa_H^{oo} = 48\%$, $\kappa_H^{w} = 2\%$								
3	Initial	9.64%	350%	46%	29.9%	15.3%	28.2%	12.9%	5.9%	6.6%
4	Today	4.81%	741%	61%	38.2%	18.%	34.8%	12.%	7.6%	11.1%
		Initial: $\kappa_H^{oo} = 28.8\%$, $\kappa_H^{w} = 1.2\%$; Today: $\kappa_H^{oo} = 42\%$, $\kappa_H^{w} = 1.75\%$								
5	Initial	9.97%	350%	48%	30.2%	16.%	28.4%	13.3%	5.8%	6.6%
6	Today	5.94%	592%	54%	33.9%	15.5%	31.4%	11.4%	6.9%	9.1%
		$\kappa_H^{oo}/\kappa_N^{oo} = \kappa_H^{w}/\kappa_N^{w} = 120\%$; Initial: $\kappa_N^{oo} = 24\%$, Today: $\kappa_N^{oo} = 34\%$								
7	Initial	9.99%	349%	48%	30.2%	15.9%	28.4%	13.3%	5.8%	6.6%
8	Today	5.3%	604%	52%	33.8%	14.4%	31.5%	10.5%	6.8%	9.%

Note: The benchmark specification of the four-groups-model comes from the lower part of Table 2. The first alternative scenario uses an increase by 100% in the housing stock of outright owners, i.e. $\kappa_H^{oo} = 48\%$, $\kappa_H^{w} = 2\%$ and $s_B^w = 0.31$. For the second scenario I assume an increase from $\kappa_H^{oo} = 28.8\%$, $\kappa_H^{w} = 1.2\%$ to $\kappa_H^{oo} = 42\%$, $\kappa_H^{w} = 1.75\%$ together with $s_B = 0.0076$ and $s_B^w = 0.084$ (initial) and $s_B^w = 0.19$ (today). In the third scenario $\kappa_H^{oo} = 28.8\%$, $\kappa_H^{w} = 1.2\%$ (in both the initial and current situation) while the share of the outright owners increases from $\kappa_N^{oo} = 24\%$ to $\kappa_N^{oo} = 34\%$ and $s_B = 0.0076$ and $s_B^w = 0.084$ (initial) and $s_B^w = 0.24$ (today).

Index of Working Papers:

January 13, 2021	Maximilian Böck, Martin Feldkircher, Burkhard Raunig	233	A View from Outside: Sovereign CDS Volatility as an Indicator of Economic Uncertainty
May 20, 2021	Burkhard Raunig	234	Economic Policy Uncertainty and Stock Market Volatility: A Causality Check
July 8, 2021	Thomas Breuer, Martin Summer, Branko Uroševic	235	Bank Solvency Stress Tests with Fire Sales
December 14, 2021	Michael Sigmund, Kevin Zimmermann	236	Determinants of Contingent Convertible Bond Coupon Rates of Banks: An Empirical Analysis
February 14, 2022	Elisabeth Beckmann, Christa Hainz, Sarah Reiter	237	Third-Party Loan Guarantees: Measuring Literacy and its Effect on Financial Decisions
February 16, 2022	Markus Knell, Reinhard Koman	238	Pension Entitlements and Net Wealth in Austria
May 9, 2022	Nicolás Albacete, Pirmin Fessler, Peter Lindner	239	The Wealth Distribution and Redistributive Preferences: Evidence from a Randomized Survey Experiment
June 20, 2022	Erwan Gautier, Cristina Conflitti, Riemer P. Faber, Brian Fabo, Ludmila Fadejeva, Valentin Jouvanceau, Jan-Oliver Menz, Teresa Messner, Pavlos Petroulas, Pau Roldan-Blanco, Fabio Rumler, Sergio Santoro, Elisabeth Wieland, Hélène Zimmer	240	New Facts on Consumer Price Rigidity in the Euro Area
June 29, 2022	Svetlana Abramova, Rainer Böhme, Helmut Elsinger, Helmut Stix	241	What can CBDC designers learn from asking potential users? Results from a survey of Austrian residents

July 1, 2022	Marcel Barmeier	242	The new normal: bank lending and negative interest rates in Austria
July 14, 2022	Pavel Ciaian, Andrej Cupak, Pirmin Fessler, d'Artis Kancs	243	Environmental-Social-Governance Preferences and Investments in Crypto-Assets
October 18, 2022	Burkhard Raunig, Michael Sigmund	244	The ECB Single Supervisory Mechanism: Effects on Bank Performance and Capital Requirements
April 5, 2023	Norbert Ernst, Michael Sigmund	245	Are zombie firms really contagious?
May 8, 2023	Richard Sellner, Nico Pintar, Norbert Ernst	246	Resource Misallocation and TFP Gap Development in Austria
September 5, 2023	Katharina Allinger, Fabio Rumler	247	Inflation Expectations in CESEE: The Role of Sentiment and Experiences
October 16, 2023	Pietro Saggese, Esther Segalla, Michael Sigmund, Burkhard Raunig, Felix Zangerl, Bernhard Haslhofer	248	Assessing the Solvency of Virtual Asset Service Providers: Are Current Standards Sufficient?
October 20, 2023	Pirmin Fessler, Severin Rapp	249	The subjective wealth distribution: How it arises and why it matters to inform policy?
October 27, 2023	Burkhard Raunig, Michael Sigmund	250	Watching over 21,000 Billion Euros: Does the ECB Single Supervisory Mechanism Affect Bank Competition in the Euro Area?
December 5, 2023	Markus Knell	251	Housing and the secular decline in real interest rates