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Financial Networks, Cross Holdings, and Limited Liability

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Editorial

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Helmut Elsinger [*]

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1. Introduction

Banks are connected with each other through a complicated network of financial claims and obligations. The value of these claims depends on the financial health of the obligor who himself might be an obligee such that his financial health depends on his obligors. Through these linkages financial distress of one bank might draw other banks into default and thereby create a domino effect of bank failures – a systemic crisis. Under the labels of *systemic risk* and *contagion* this problem has gained considerable attention in the literature on financial intermediaries (Allen and Gale (2000), Giesecke and Weber (2004), Rochet and Tirole (1997), and Shin (2006)).

Modeling correlated defaults is not only an issue in the banking literature but also in the literature on the valuation of complex portfolio credit derivatives such as collateralized debt obligations (CDOs). The value of the different CDO tranches depends crucially on the joint distribution of default of the underlying collateral securities. The linkages between these securities (obligors) are not modeled explicitly but via the assumption that the default intensities of these securities are correlated (Duffie and Gârleanu (2001), Longstaff and Rajan (2008), and Errais et al. (2009)).

According to Boyd et al. (2005) the social costs of a systemic crisis range from 60% to 300% of GDP. The prevention of a breakdown of financial intermediation is of vital interest for central banks and regulatory authorities. To assess the probability and severity of such a cascade of defaults caused by interbank lending a large number of central banks perform counterfactual simulations.[1] In a first step bilateral credit exposures in the interbank market are determined.[2] Given these exposures there are two approaches in the literature to simulate contagion. The first approach introduced by Furfine (2003) assumes that a particular bank is not able to honor its obligations. All creditors of this bank lose an exogenously specified fraction of their claims against this bank (loss given default). If the losses of an affected bank exceed its capital, the bank is in default, too. In the next round the creditors of all defaulting banks lose a fraction of their claims against these banks. Again these losses are compared to the capital available. The procedure is iterated until no additional bank goes bankrupt. If this simulation is performed for each bank, it allows to determine which banks are systemically relevant in the sense that they trigger contagious defaults. Yet, the approach is not able to assess the probability that a particular bank defaults. Moreover, loss given default is not endogenous but exogenously given.

[1] The different models used are discussed thoroughly in Upper (2007).

[2] Typically these exposures are not readily available. They have to be estimated from aggregate data. Mistrulli (2006) discusses the consequences of estimation errors.

The second simulation approach is based on a model developed by Eisenberg and Noe (2001). In their framework banks are not only linked with each other via the interbank market but are also endowed with exogenous income. Under the assumption of limited liability of equity and absolute priority of debt Eisenberg and Noe (2001) show that for a given level of exogenous income the equity values of the banks, default and loss given default can be determined endogenously. Elsinger et al. (2006a) introduced this approach to the contagion literature by applying it to the Austrian banking system. They take the net position of all non-interbank related parts of the balance sheet as exogenous income. Using standard risk management techniques they simulate changes in the value of this exogenous income for all banks in the system simultaneously. Given such a scenario the interbank market is cleared and the equity values of the banks and the values of the interbank debt contracts are determined. The advantage of this approach is that it not only allows to identify systemically important banks but also allows to assess the probability of default and the level of loss given default endogenously. Yet, two important features commonly observed in real world networks are not included in this framework. Firstly, cross shareholdings between the banks are not explicitly modeled. The value of such holdings has to be treated as a part of exogenous income. This leads to inconsistencies. The (simulated) value of the holding will not equal the value after the clearing procedure. Secondly, the seniority structure of debt is not modeled. It is assumed that all debt in the interbank market is of the same seniority and that it is junior relative to all other debt claims against the banks.

The aim of this paper is to extend the work of Eisenberg and Noe (2001) by taking cross holdings and a detailed seniority structure of debt explicitly into account. Banks are modeled as nodes in a network which are endowed with exogenous income. They may have nominal obligations to other nodes in the network and to outside creditors. Furthermore the nodes may hold equity shares of other nodes. I assume equity has limited liability, absolute priority of debt, proportional rationing in case of default, and that there is no subset of nodes where each node in the subset is owned entirely by the other nodes in the subset. Given these assumptions neither equity nor debt values are necessarily unique. Yet, banks with non-unique equity values have to be entirely owned either by other banks with non-unique equity values or zero equity value. If the debt payments of a bank are non-unique it has to hold that all claimants are themselves banks with either non-unique debt payments or non-unique equity value. Hence, the values of debt and equity claims that are held by outside investors are unique. Moreover, given there are no bankruptcy costs it never pays for outside investors to bail out defaulting banks.

I develop a solution algorithm for the clearing problem which lacks the elegance of the fictitious default algorithm developed by Eisenberg and Noe (2001) but which is applicable under weaker assumptions on the structure of the network. This algorithm allows to incorporate cross holdings and a detailed seniority structure.

In a financial network without cross holdings Eisenberg and Noe (2001) show that equity values are convex and debt values are concave in the exogenous income. Using a simple example I show that is not true as soon as cross holdings are included.

The paper is organized as follows. In Section 2 a financial network with cross holdings is presented and the main concepts are developed. In Section 3 I show that a clearing vector exists and characterize networks for which the clearing vector is not unique. In Section 4 a solution algorithm is presented. I discuss the comparative statics in Section 5. In Section 6 the model is augmented with a detailed seniority structure. It is shown that the main results remain valid. FI show that given there are no bankruptcy costs it never pays for outside investors to bail out defaulting nodes. Finally, Section 7 concludes the paper.

2. The Model

Consider an economy populated by n banks constituting a *financial network*. Each of these banks is endowed with an exogenous income $e_i \in \mathbb{R}$ which may be negative. Without a detailed priority structure of debt e_i may be interpreted as operating income minus all liabilities except the most junior. Ruling out that e_i might be negative would be equivalent to assuming that all liabilities except the most junior are always repaid in full.[3] Any bank may hold shares of companies outside the network. The value of these holdings is not determined endogenously. It is included in e.

Banks may have nominal obligations to other banks in the network. The structure of these liabilities is represented by an $n \times n$ matrix L where L_{ij} represents the nominal obligation of bank i to bank j. These liabilities are nonnegative and the diagonal elements of L are zero as banks are not allowed to hold liabilities against themselves. Liabilities to creditors outside the network are denoted by $D_i \geq 0$. Furthermore, banks may hold shares of other banks which are denoted by the matrix $\Theta \in [0, 1]^{n\times n}$ where Θ_{ij} is the share of bank i held by bank j. A bank may be among the shareholders of its own shares ($\Theta_{ii} > 0$).

Suppose there are two banks, A and B. Bank A has an exogenous income of 1, no outstanding debt, and owns bank B entirely. On the other hand bank B has an

[3] In Section 6 the framework is extended to deal with different seniority classes.

exogenous income of 2, no debt and owns bank A. The equity value of A equals 1 plus the equity value of B. B's equity value equals 2 plus the equity value of A. The only solution would be that both banks have an equity value of infinity. In this example ownership is not well defined irrespective whether there is limited or unlimited liability. To make sure that ownership is well defined it suffices to assume that there is no group of banks where *each* bank is completely owned by other banks in that group. In particular, a bank must not own itself entirely ($\Theta_{ii} < 1$).

Assumption 1. *There exists no subset $\mathcal{I} \subset \{1, \ldots, n\}$ such that*

$$\sum_{j \in \mathcal{I}} \Theta_{ij} = 1 \quad \textit{for all} \quad i \in \mathcal{I}.$$

with $\Theta \in [0,1]^{n \times n}$ and $\Theta \vec{1} \leq \vec{1}$ where $\vec{1}$ is an $n \times 1$ vector of ones. Θ is called a holding matrix if it fulfills this assumption.

A bank is defined to be in default whenever exogenous income plus the amounts received from other nodes plus the value of the holdings are insufficient to cover the bank's nominal liabilities.[4] Throughout the paper I assume that bank defaults do not change the prices outside of the network, i.e. e is independent of defaults and exogenous. If default changes prices due to e.g. fire sales, the story is different. Only in the special case where defaults unequivocally decrease e the main results still hold.

Exogenous income e may be interpreted as a multidimensional random variable. For each draw of e the system is cleared. All the results in the sequel are conditional on a particular draw. Moreover, in principle the entire economy can be represented by such a network with the limitation that the behavior of the nodes is not modeled.

In case of default the clearing procedure has to respect three criteria:

1. limited liability: which requires that the total payments made by a node must never exceed the sum of exogenous income, payments received from other nodes, and the value of the holdings,

2. priority of debt claims: which requires that stockholders receive nothing unless the bank is able to pay off all of its outstanding debt completely, and

3. proportionality: which requires that in case of default all claimant nodes are paid off in proportion to the size of their claims on firm assets.

[4] A bank is in default if liabilities exceed assets. Using a violation of capital requirements as default threshold does not change the main results.

To operationalize proportionality let $\bar{p}_i$ be the total nominal obligations of node i, i.e.

$$\bar{p}_i = \sum_{j=1}^{n} L_{ij} + D_i$$

and define the proportionality matrix Π by

$$\Pi_{ij} = \begin{cases} \frac{L_{ij}}{\bar{p}_i} & \text{if } \bar{p}_i > 0 \\ 0 & \text{otherwise} \end{cases}$$

Evidently, it has to hold that $\Pi \cdot \vec{1} \leq \vec{1}$.

To simplify notation I define for any two vectors $x, y \in \mathbb{R}^n$ the lattice operations

$$x \wedge y := (min(x_1, y_1), \cdots, min(x_n, y_n))$$
$$x \vee y := (max(x_1, y_1), \cdots, max(x_n, y_n))$$

and I introduce a matrix notation that allows to describe summation over index sets by matrix multiplication conveniently.

Definition 1. *Let y and x be $n \times 1$ vectors. Then $\Lambda := diag(y \geq x)$ is an $n \times n$ diagonal matrix where $\Lambda_{ii} = 1$ if $y_i \geq x_i$ and $\Lambda_{ii} = 0$ otherwise. $diag(y > x)$, $diag(y \leq x)$, $diag(y < x)$, $diag(y \neq x)$, and $diag(y = x)$ are defined analogously.*

Let $p = (p_1, \ldots, p_n)' \in \mathbb{R}^n_+$ be a vector of payments made by banks to their interbank and non interbank creditors. To define the equity values V of the banks assume for a moment that these values are exogenously given ($V \geq \vec{0}$) and define the map

$$\Psi^1(V, p, e, \Pi, \Theta) = [e + \Pi' p - p + \Theta' V] \vee \vec{0} \tag{1}$$

where $\vec{0}$ denotes the $n \times 1$ dimensional zero vector. Ψ^1 returns the values of the nodes given V and p. A necessary condition for V to be a vector of equity values is that V is a fixed point, $V^*(p)$, of $\Psi^1(\cdot; p, e, \Pi, \Theta) : \mathbb{R}^n_+ \to \mathbb{R}^n_+$, i.e.

$$V^*(p) = [e + \Pi' p - p + \Theta' V^*(p)] \vee \vec{0}. \tag{2}$$

If Θ is a holding matrix, Lemma 4 in the Appendix establishes that $V^*(p)$ is unique for any p. However, for arbitrary p it is possible that $V_i^*(p) > 0$ and $p_i < \bar{p}_i$. Absolute priority would not hold. Given p the amount available for bank i to pay off its debt equals $e_i + \sum_{j=1}^{n} \Pi_{ji} p_j + \sum_{j=1}^{n} \Theta_{ji} V_j^*(p)$. If this amount is less than zero, bank i will pay nothing due to limited liability. If this amount is larger than the liabilities ($\bar{p}_i$), bank i pays off debt completely. If the amount available is in the range from zero to $\bar{p}_i$, it is distributed proportionally amongst the debtholders. A vector of payments p^* respects the clearing criteria if

$$p_i^* = \begin{cases} 0 & \text{for} \quad e_i + \sum\limits_{j=1}^{n} \left(\Pi_{ji} p_j^* + \Theta_{ji} V_j^*(p^*)\right) \leq 0 \\ e_i + \sum\limits_{j=1}^{n} \left(\Pi_{ji} p_j^* + \Theta_{ji} V_j^*(p^*)\right) & \text{for} \quad 0 \leq e_i + \sum\limits_{j=1}^{n} \left(\Pi_{ji} p_j^* + \Theta_{ji} V_j^*(p^*)\right) \leq \bar{p}_i \\ \bar{p}_i & \text{for} \quad \bar{p}_i \leq e_i + \sum\limits_{j=1}^{n} \left(\Pi_{ji} p_j^* + \Theta_{ji} V_j^*(p^*)\right). \end{cases}$$

For such a payment vector $V_i^*(p^*) > 0$ implies that $p_i^* = \bar{p}_i$. $V^*(p^*)$ is a vector of equity values consistent with limited liability, absolute priority, and proportional rationing in the case of default. As $V^*(p)$ is unique for arbitrary p a vector of payments p^* that respects the clearing criteria can be defined unambiguously.

Definition 2. *A vector $p^* \in [\vec{0}, \bar{p}]$ is a* **clearing payment vector** *if*

$$p^* = \left\{ \left[e + \Pi' p^* + \Theta' V^*(p^*) \right] \vee \vec{0} \right\} \wedge \bar{p} \tag{3}$$

where $V^*(p^*)$ is the unique fixed point of $\Psi^1(\cdot; p^*, e, \Pi, \Theta)$.

Alternatively, a clearing vector p^* can be characterized as a fixed point of the map $\Phi^1(\cdot; \Pi, \bar{p}, e, \Theta) : [\vec{0}, \bar{p}] \to [\vec{0}, \bar{p}]$ defined by

$$\Phi^1(p; \Pi, \bar{p}, e, \Theta) = \left\{ \left[e + \Pi' p + \Theta' V^*(p) \right] \vee \vec{0} \right\} \wedge \bar{p} \tag{4}$$

In the framework without cross holdings ($\Theta = \mathbf{0}_{n,n}$) Eisenberg and Noe (2001) show that such a clearing vector exists. Furthermore, they are able to specify sufficient conditions to guarantee uniqueness.

3. Existence and Uniqueness of a Clearing Payment Vector

The definition of the equity value as stated in Equation (2) has to be adapted to prove the existence of a clearing vector. Suppose – for a moment – that ($\Theta = \mathbf{0}_{n,n}$). In this case the value of bank i is given by $V_i^*(p^*) = max(e_i + \sum_{j=1}^{n} \Pi_{ji} p_j^* - p_i^*, 0)$ where p^* is a clearing vector. $V_i^*(p^*) > 0$ implies that $p_i^* = \bar{p}_i$. The value of bank i may therefore be written as $max(W_i^*(p^*), 0)$ where $W_i^*(p^*) = e_i + \sum_{j=1}^{n} \Pi_{ji} p_j^* - \bar{p}_i$. If Θ is arbitrary, this translates into

$$W^*(p) = [e + \Pi' p - \bar{p}] + \Theta'(W^*(p) \vee \vec{0}). \tag{5}$$

$W^*(p)$ is the vector of node values under the assumption of limited liability for the cross holdings. Yet, $W^*(p)$ is not the equity value of the banks as $W^*(p)$ is not necessarily nonnegative. $W^*(p)$ may be defined as a fixed point of the function of $\Psi^2(\cdot; p, \bar{p}, e, \Pi, \Theta) : \mathbb{R}^n \to \mathbb{R}^n$, given by

$$\Psi^2(W, p, \bar{p}, e, \Pi, \Theta) = e + \Pi' p - \bar{p} + \Theta'(W \vee \vec{0}). \tag{6}$$

Lemma 5 in the Appendix establishes that $W^*(p)$ is unique for arbitrary p and that $W^*(p)$ is increasing in p. The definition of a clearing vector has to be adjusted to this alternative definition of equity values by substituting $W^*(p) \vee \vec{0}$ for $V^*(p)$. I define the map $\Phi^2(p; \Pi, \bar{p}, e, \Theta) : [\vec{0}, \bar{p}] \to [\vec{0}, \bar{p}]$ by

$$\Phi^2(p; \Pi, \bar{p}, e, \Theta) = \left\{ \left[e + \Pi' p + \Theta'(W^*(p) \vee \vec{0}) \right] \vee \vec{0} \right\} \wedge \bar{p} = \left\{ [W^*(p) + \bar{p}] \vee \vec{0} \right\} \wedge \bar{p} \tag{7}$$

where $W^*(p)$ is the unique fixed point of Ψ^2. If p^* is a fixed point of $\Phi^2(p)$ then it holds that $W_i^*(p^*) \geq 0$ is equivalent to $p_i^* = \bar{p}_i$. So, we may call p^* a clearing vector and $W^*(p^*) \vee \vec{0}$ the vector of equity values.

Theorem 1. *Let $\hat{p} \in [\vec{0}, \bar{p}]$ be a (super)solution of $\Phi^1(p)$, i.e. $\hat{p} \geq \Phi^1(\hat{p}; \Pi, \bar{p}, e, \Theta)$.*[5] *Then $\hat{p}$ is a (super)solution of $\Phi^2(p)$ with $W^*(\hat{p}) = e + \Pi' \hat{p} - \bar{p} + \Theta' V^*(\hat{p})$. If $\hat{p} \in [\vec{0}, \bar{p}]$ is a (super)solution of $\Phi^2(p)$ then $\hat{p}$ is a (super)solution of $\Phi^1(p)$ with $V^*(\hat{p}) = (W^*(\hat{p}) \vee \vec{0})$.*

Proof. I prove the assertion for the case of supersolutions. The proof for solutions is analogous. Suppose that $\hat{p}$ is a supersolution of $\Phi^1(p)$, i.e. $\hat{p} \geq \Phi^1(\hat{p})$. Define $X = e + \Pi' \hat{p} - \bar{p} + \Theta' V^*(\hat{p})$. We have to show that X is a solution to Equation (5), i.e. $V^*(\hat{p}) = (X \vee \vec{0})$. By construction $V^*(\hat{p}) \geq X$. If the equity value of bank i is larger than 0, it has to hold that $e_i + \sum_{j=1}^n \Pi_{ji} \hat{p}_j + \sum_{j=1}^n \Theta_{ji} V_j^*(\hat{p}) > \hat{p}_i$. As $\hat{p}$ is a supersolution of $\Phi^1(p)$ this implies that $\hat{p}_i = \bar{p}_i$ and $V_i^*(\hat{p}) = X_i$. Therefore, $(X \vee \vec{0}) = V(\hat{p})$. This yields $X = e + \Pi' \hat{p} - \bar{p} + \Theta(X \vee \vec{0})$ and $\hat{p} \geq \Phi^2(\hat{p})$.

Now suppose $\hat{p}$ is a supersolution of $\Phi^2(p)$. If $W_i^*(\hat{p}) \geq 0$ then (7) implies that $\bar{p}_i \geq \hat{p}_i \geq \Phi_i^2(\hat{p}) = \bar{p}_i$ and $W_i^*(\hat{p}) = e_i + \sum_{j=1}^n \Pi_{ji} \hat{p}_j - \hat{p}_i + \sum_{j=1}^n \Theta_{ji}(W_j^*(\hat{p}) \vee 0)$. If, on the other hand, $W_i^*(\hat{p}) < 0$ then we get $\bar{p}_i > \Phi_i^2(\hat{p}) \geq e_i + \sum_{j=1}^n \Pi_{ji} \hat{p}_j + \sum_{j=1}^n \Theta_{ji}(W_j^*(\hat{p}) \vee 0)$. The vector $\hat{p}$ is a supersolution of Φ^2 and therefore

$$0 \geq \Phi_i^2(\hat{p}) - \hat{p}_i \geq e_i + \sum_{j=1}^n \Pi_{ji} \hat{p}_j - \hat{p}_i + \sum_{j=1}^n \Theta_{ji}(W_j^*(\hat{p}) \vee 0).$$

So, $W^*(\hat{p}) \vee \vec{0} = [e + \Pi' \hat{p} - \hat{p} + \Theta'(W^*(\hat{p}) \vee \vec{0})] \vee \vec{0}$ and $X = W^*(\hat{p}) \vee \vec{0}$ solves (2). Plugging X into (4) shows that $\bar{p}$ is a supersolution of Φ^1. $\square$

[5] $\hat{p}$ is a supersolution if the proposed payments $\hat{p}$ exceed the payments required by limited liability and absolute priority under the assumption that the proposed payments are actually paid, i.e. $\Phi^1(\hat{p}; \Pi, \bar{p}, e, \Theta)$.

Any fixed point of $\Phi^2(p)$ is a fixed point of $\Phi^1(p)$ and vice versa. To prove that a clearing vector exists it suffices to show that $\Phi^2(p)$ has a fixed point. Note that by construction $\Phi^2(\vec{0}) \geq \vec{0}$ and $\Phi^2(\bar{p}) \leq \bar{p}$. The Tarski fixed point theorem guarantees that there exists a least and a greatest fixed point for $\Phi^2(p)$ if $\Phi^2(p)$ is a monotone increasing function on the complete lattice $[0, \bar{p}]$. Lemma 5 shows that $W^*(p)$ and thereby $\Phi^2(p)$ are increasing in p.

Theorem 2. *There exists a greatest (p^+) and a least (p^-) clearing vector.*

If the clearing vector is not unique it might happen that the equity values of the banks are different for different clearing vectors. In particular it could happen that a bank that is in default at p^- might have a positive equity value at p^+. Eisenberg and Noe (2001) show that for $\Theta = \mathbf{0}_{n,n}$ the equity values of the nodes do not depend on the chosen clearing vector. A bank defaulting at p^- might not default at p^+. Yet, the equity value at p^+ has to be zero and the bank is only just solvent. In the more general framework under consideration in this paper the situation is more complicated as is illustrated by the following example.

Example 1. *Assume that the network is characterized by the following parameters.*

$$e = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \quad \Pi = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \quad \bar{p} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}, \quad \Theta = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$$

It is easy to check that any $p(\lambda) = (1, \lambda)'$ for $\lambda \in [0,1]$ is a clearing vector with corresponding equity value $V^*(\lambda) = (\lambda, 0)'$. The equity value of bank 1 is not unique. The reason is that any amount $\lambda \in [0,1]$ paid by bank 2 is received entirely by bank 1 as it is the only obligee of bank 2. Bank 1 is able to cover all liabilities by exogenous income. Any additional payments received increase the equity value by the very same amount. But bank 1 is entirely owned by bank 2. Hence, the value of the holdings of bank 2 increases by λ and the initial payment of λ is affordable.

The existence of multiple clearing vectors is extremely sensitive to the chosen parameter values. If Π_{21} or Θ_{12} or both are smaller than 1, the unique clearing payment will be $(1,0)'$ with corresponding equity values $(0,0)'$. Moreover, if the aggregate exogenous income is not equal to 1, the clearing vector is unique, too.

Suppose that p^2 and p^1 are two distinct clearing vectors such that $p^2 \geq p^1$. Denote the corresponding equity values by V^2 and V^1. Lemma 5 implies that $W^2 \geq W^1$ and this in turn implies that $V^2 \geq V^1$. Equity values are increasing in the clearing vectors. Let $\Lambda^1 = diag(V^2 > V^1)$, $\Lambda^2 = diag(p^2 > p^1)$, and $\Lambda = \mathbf{I} - (\mathbf{I} - \Lambda^1)(\mathbf{I} - \Lambda^2)$. Λ^1

characterizes the set of banks that have different equity values under the different clearing vectors. Λ^2 characterizes the banks with different clearing vectors and Λ describes the 'union' of the banks that have either a non-unique equity value or a non-unique clearing payment.[6] If bank i is in Λ then $p_i^2 > 0$ and $V_i^2 = e_i + \sum_{j=1}^n \Pi_{ji} p_j^2 - p_i^2 + \sum_{j=1}^n \Theta_{ji} V_j^2$. Subtracting V^1 from V^2 yields $\Lambda(V^2 - V^1) \leq \Lambda[(\Pi' - \mathbf{I})(p^2 - p^1) + \Theta'(V^2 - V^1)]$. Note that $V^2 - V^1 = \Lambda^1(V^2 - V^1) = \Lambda(V^2 - V^1)$ and $p^2 - p^1 = \Lambda^2(p^2 - p^1) = \Lambda(p^2 - p^1)$. After rearranging and premultiplying by $\vec{1}'$ we get

$$\vec{1}'\Lambda(\mathbf{I} - \Theta')\Lambda^1(V^2 - V^1) \leq \vec{1}'\Lambda(\Pi' - \mathbf{I})\Lambda^2(p^2 - p^1).$$

The left hand side is larger than or equal to zero whereas the right hand side is smaller than or equal to zero. For the inequality to hold both sides have to equal zero and the inequality turns into an equality.

$$\vec{1}'\Lambda(\mathbf{I} - \Theta')\Lambda^1(V^2 - V^1) = \vec{1}'\Lambda(\Pi' - \mathbf{I})\Lambda^2(p^2 - p^1) = 0 \tag{8}$$

The left hand side of (8) implies that $\sum_{j=1}^n \Lambda_{jj}^1 (1 - \sum_{i=1}^n \Theta_{ji}\Lambda_{ii})\Lambda_{jj}^1 (V_j^2 - V_j^1) = 0$. If bank j has a non–unique equity value then $\Lambda_{jj}^1 = 1$ and $\Lambda_{jj}^1(V_j^2 - V_j^1) > 0$. Bank j has to be owned entirely by banks in Λ as $\sum_{i=1}^n \Theta_{ji}\Lambda_{ii}$ has to equal 1. Analogously, if bank j has a non–unique clearing payment, $\sum_{i=1}^n \Pi_{ji}\Lambda_{ii}$ has to equal 1. All obligees of bank j have to belong to Λ.

Theorem 3. *The clearing vector and the equity values are unique if there is no subset $\mathcal{I}$ of banks such that for all $i \in \mathcal{I}$ either*

$$\textstyle\sum_{j\in\mathcal{I}} \Theta_{ij} = 1 \quad \text{or} \quad \sum_{j\in\mathcal{I}} \Pi_{ij} = 1 \tag{9}$$

For $\Theta = \mathbf{0}_{n,n}$ the clearing vector is unique if Π fulfills Assumption 1.

Theorem 3 characterizes a sufficient but not necessary condition for uniqueness. Even if the network structure allows for multiple clearing vectors, it is possible that the clearing vector is unique. For very large exogenous income ($e > \bar{p}$) $\bar{p}$ is the only possible clearing vector. To improve this condition on e, let Λ, Λ^1, and Λ^2 be defined as before and note that $\Lambda V^2 = \Lambda(e + (\Pi' - \mathbf{I})p^2 + \Theta' V^2)$. Rearranging and premultiplying by $\vec{1}'$ yields

$$\vec{1}'\Lambda(\mathbf{I} - \Theta')\Lambda V^2 + \vec{1}'\Lambda(\mathbf{I} - \Pi')\Lambda p^2 = \vec{1}'\Lambda(e + \Pi'(\mathbf{I} - \Lambda)p^2 + \Theta'(\mathbf{I} - \Lambda)V^2).$$

Banks with non-unique clearing payment but unique equity value have to have an

[6] In a slight abuse of notation I will sometimes use "bank i is in Λ" meaning that $\Lambda_{ii} = 1$.

equity value of 0, i.e. $(\Lambda - \Lambda^1)V^2 = \vec{0}$. Together with $\vec{1}'\Lambda(\mathbf{I} - \Theta')\Lambda^1 = 0$ we get $\vec{1}'\Lambda(\mathbf{I} - \Theta')\Lambda V^2 = 0$. For banks with a unique clearing payment and non-unique equity value it has to hold that the clearing payment equals the promised payment, i.e. $(\Lambda - \Lambda^2)p^2 = (\Lambda - \Lambda^2)\bar{p}$. Using the fact that $\vec{1}'\Lambda(\mathbf{I} - \Pi')\Lambda^2 = 0$ yields

$$0 \le \vec{1}'\Lambda(\mathbf{I} - \Pi')\Lambda\bar{p} = \vec{1}'\Lambda(e + \Pi'(\mathbf{I} - \Lambda)p^2 + \Theta'(\mathbf{I} - \Lambda)V^2). \tag{10}$$

The interpretation of this relation is straightforward. The left hand side is the sum of all promised payments made by banks with non-unique payment or equity value ($\vec{1}'\Lambda\bar{p}$) minus the sum of all liabilities within these banks ($\vec{1}'\Lambda\Pi'\Lambda\bar{p}$). So the left hand side is the net obligation of banks in Λ to banks not in Λ and creditors outside the network. The right hand side is the sum of the exogenous income of all banks in Λ ($\vec{1}'\Lambda e$) plus the sum of all debt payments from banks not in Λ to banks in Λ ($\vec{1}'\Lambda\Pi'(\mathbf{I} - \Lambda)p^2$) plus the value of shares of banks not in Λ that are held by banks in Λ ($\vec{1}'\Lambda\Theta'(\mathbf{I} - \Lambda)V^2$). In the case of multiple clearing vectors the sum of the obligations of banks in Λ to banks not in Λ has to equal the aggregate exogenous income of banks in Λ plus the value of all claims of banks in Λ against banks not in Λ. An immediate consequence of (10) is that $\vec{1}'\Lambda e > \vec{1}'\Lambda(\mathbf{I} - \Pi')\Lambda\bar{p}$ precludes the existence of multiple clearing vectors.

Theorem 4. *Suppose for any subset $\mathcal{I}$ of banks with*

$$\textstyle\sum_{j\in\mathcal{I}} \Theta_{ij} = 1 \quad \textit{or} \quad \sum_{j\in\mathcal{I}} \Pi_{ij} = 1 \quad \textit{for all} \quad i \in \mathcal{I}$$

it holds that $\sum_{i\in\mathcal{I}} e_i > \sum_{i\in\mathcal{I}}(1 - \sum_{j\in\mathcal{I}} \Pi_{ij})\bar{p}_i$ then the clearing vector is unique. For $\Theta = \mathbf{0}_{n,n}$ the clearing vector is unique if $\sum_{i\in\mathcal{I}} e_i > 0$.

Proof. Only the claim for $\Theta = \mathbf{0}_{n,n}$ remains to be shown. But in this case $\sum_{j\in\mathcal{I}} \Pi_{ij} = 1$ for all $i \in \mathcal{I}$. □

It might be possible that the value of the debt and equity holdings of an outside investor depends on the chosen clearing vector. Let ϵ be the (column) vector of shares owned by an outside investor and let δ equal the investor's share in debt. Given p^* and $V^*(p^*)$ the value of the outside investors portfolio equals $\epsilon' V^*(p^*) + \delta' p^*$. Clearly, it has to hold that the equity share in bank i owned by the outside investor (ϵ_i) has to be less than or equal to 1 minus the shares held by banks in the network, i.e. $\epsilon_i \le 1 - \sum_{j=1}^{n} \Theta_{ij}$. So, $\epsilon \le \vec{1}'(\mathbf{I} - \Theta')$ and analogously $\delta \le \vec{1}'(\mathbf{I} - \Pi')$. If the equity value of bank i is non–unique then the bank is entirely owned by other banks, i.e. $1 - \sum_{j=1}^{n} \Theta_{ij} = 0$ and $\epsilon_i = 0$. If the clearing payment of bank i is non–unique then the only obligees are other banks, i.e. $1-\sum_{j=1}^{n}\Pi_{ij}=0$ and $\delta_i=0$. For two different clearing vectors p^1 and p^2 and corresponding equity values V^1 and V^2 it has to hold that $\epsilon'(V^2-V^1)=\delta'(p^2-p^1)=0$.

Theorem 5. *The value of an outside investor's portfolio is independent of the chosen clearing vector. In particular, for arbitrary clearing vectors p^1 and p^2 and corresponding equity values V^1 and V^2 it holds that $\vec{1}'(\mathbf{I}-\Theta)(V^2-V^1)=\vec{1}'(\mathbf{I}-\Pi')(p^2-p^1)=0$.*

To model bankruptcy costs, the framework needs to be adapted. Denote the vector of bankruptcy costs by $b(p)$ and assume that $b(p)$ is decreasing in p. The simplest type of bankruptcy costs would be such that a bank i that is in default loses a specified exogenous amount $c_i>0$, i.e. $b_i(\bar{p}_i)=0$ and $b_i(p_i)=c_i$ for all $p_i<\bar{p}_i$. A sufficient condition for a clearing vector to exist is that $W^*(p)$ is increasing in p. With bankruptcy costs $W^*(p)$ has to be defined as a fixed point of

$$\hat{\Psi}^2(W,p,\bar{p},e,\Pi,\Theta)=e(p)+\Pi'p-\bar{p}+\Theta'(W\vee\vec{0}) \tag{11}$$

where $e(p)=e-b(p)$ and is increasing in p. $W^*(p)$ is unique and increasing in p by Lemma 5 in the Appendix. Applying the Tarski fixed point theorem to

$$\hat{\Phi}^2(p;\Pi,\bar{p},e,\Theta)=\left\{\left[e(p)+\Pi'p+\Theta'(W^*(p)\vee\vec{0})\right]\vee\vec{0}\right\}\wedge\bar{p} \tag{12}$$

yields that a greatest and a least clearing vector exist. If e depends on p, the conditions stated in Theorems 3 and 4 do not suffice to guarantee a unique clearing vector. Moreover, the value of an outside investor's portfolio will depend on the chosen clearing vector. To bail out defaulting banks may be profitable.

4. Calculating a Clearing Vector

For the case $\Theta=\mathbf{0}_{n,n}$ Eisenberg and Noe (2001) develop an extremely elegant algorithm to calculate clearing vectors, the fictitious default algorithm. It has the nice feature that it reveals a sequence of defaults. In the first round of the algorithm it is assumed that the payments made equal the promised payments $\bar{p}$. Banks that are unable to meet their obligations are determined. These banks default even if all of their interbank claims are honored. In the next step the payments of these defaulting banks are adjusted such that they are in line with limited liability. If there are no additional defaults the iteration stops. If there are further defaults the procedure is continued. The important point is that the algorithm allows to distinguish between defaults that are directly related to adverse economic situations – exogenous income – and defaults that are caused by the defaults of other banks. The fictitious default algorithm works for $e > \vec{0}$. For $e \in \mathbb{R}^n$ the algorithm might break down as is demonstrated in the next example.

Example 2. *To calculate a clearing vector for the case where $\Theta = \mathbf{0}_{n,n}$ Eisenberg and Noe (2001) propose the following iterative procedure. Let $\Lambda(p) = diag((\Pi' p + e) < \bar{p})$ and define the map $p \to FF_{\hat{p}}(p)$ as follows:*

$$FF_{\hat{p}}(p) \equiv \Lambda(\hat{p})[\Pi'(\Lambda(\hat{p})p + (\mathbf{I} - \Lambda(\hat{p}))\bar{p} + e] + (\mathbf{I} - \Lambda(\hat{p}))\bar{p}$$

This map returns for all nodes not defaulting under $\hat{p}$ the required payment $\bar{p}$. For all other nodes it returns the node's value assuming that non defaulting nodes pay $\bar{p}$ and defaulting nodes pay p. Under suitable restrictions this map has a unique fixed point which is denoted by $f(\hat{p})$. Note that the equation for the fixed point

$$f(\hat{p}) = \Lambda(\hat{p})[\Pi'(\Lambda(\hat{p})f(\hat{p}) + (\mathbf{I} - \Lambda(\hat{p}))\bar{p} + e] + (\mathbf{I} - \Lambda(\hat{p}))\bar{p}$$

can actually be written quite compactly as

$$[\mathbf{I} - \Lambda(\hat{p})\Pi'\Lambda(\hat{p})](f(\hat{p}) - \bar{p}) = \Lambda(\hat{p})(e + \Pi'\bar{p} - \bar{p}). \tag{13}$$

Premultiplying by $(\mathbf{I} - \Lambda(\hat{p}))$ yields

$$(\mathbf{I} - \Lambda(\hat{p}))(f(\hat{p}) - \bar{p}) = \vec{0}.$$

For banks that do not default $\Lambda_{ii}(\hat{p}) = 0$ and $f_i(\hat{p}) = \bar{p}_i$. Premultiplying (13) by $\Lambda(\hat{p})$ gives

$$\Lambda(\hat{p})(\mathbf{I} - \Pi')\Lambda(\hat{p})(f(\hat{p}) - \bar{p}) = \Lambda(\hat{p})(e + \Pi'\bar{p} - \bar{p}).$$

The ijth entry of $\Lambda(\hat{p})(\mathbf{I} - \Pi')\Lambda(\hat{p})$ is zero unless $\Lambda_{ii}(\hat{p}) = \Lambda_{jj}(\hat{p}) = 1$. To calculate the fixed point, it suffices to consider the subsystem (submatrix) of defaulting nodes. The original system of equations can be chopped up into two independent systems. This is a major advantage if the number of nodes is large and default is a rare event.

Eisenberg and Noe (2001) show that under the assumption that $e > \vec{0}$ (and $\Theta = \mathbf{0}_{n,n}$) the sequence of payment vectors $p^0 = \bar{p}$, $p^i = f(p^{i-1})$ decreases to a clearing vector in at most n iterations. The assumption that $e > \vec{0}$ is essential as is illustrated by the following example.

$$e = \begin{pmatrix} 1 \\ \frac{3}{4} \\ -\frac{9}{8} \end{pmatrix}, \quad \Pi = \begin{pmatrix} 0 & 0 & 0 \\ \frac{1}{2} & 0 & \frac{1}{2} \\ \frac{1}{4} & \frac{3}{4} & 0 \end{pmatrix}, \quad \bar{p} = \begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix}$$

Setting $p^0 = \bar{p}$ yields

$$e + \Pi' p^0 = \begin{pmatrix} \frac{9}{4} \\ \frac{3}{2} \\ -\frac{1}{8} \end{pmatrix}, \quad \Lambda(p^0) = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \textit{and} \quad p^1 = f(p^0) = \begin{pmatrix} 1 \\ -\frac{3}{20} \\ -\frac{6}{5} \end{pmatrix}.$$

Hence, p^1 is not feasible and the algorithm breaks down. A possible remedy would be to use $p^i = [f(p^{i-1}) \vee \vec{0}]$. This procedure results in $p^2 = p^1 = (1, 0, 0)'$. It is easy to verify that $(1, 0, 0)'$ is not a clearing vector. The unique clearing vector for the example is given by $p^* = (1, \frac{3}{4}, 0)'$.[7]

Even though the fictitious default algorithm might not work anymore it is still possible to define a simple but admittedly less elegant iterative procedure to calculate a clearing vector.

Theorem 6. *If Θ is a holding matrix, the sequence $p^{i+1} = [(W^*(p^i) + \bar{p}) \vee \vec{0})] \wedge \bar{p}$ started at $p^0 = \bar{p}$ is well defined, decreasing, and converges to the largest clearing vector p^+.*

Proof. p^{i+1} is well defined if $W^*(p^i)$ is well defined. This is the case as Θ is a holding matrix.

To calculate $W^*(p^i)$ let $u = e + \Pi' p^i - \bar{p}$, $W^0 = u$, $\Lambda^k = diag(W^k > \vec{0})$, and $W^{k+1} = u + \Theta' \Lambda^k W^{k+1}$. As $\Theta' \Lambda^k$ is a holding matrix Lemma 1 implies that W^{k+1} exists and is unique. By construction $\Lambda^k W^k \geq \Lambda^{k-1} W^k$. Therefore $u + \Theta' \Lambda^k W^k \geq u + \Theta' \Lambda^{k-1} W^k = W^k$. Let $y = u + \Theta' \Lambda^k W^k - W^k \geq \vec{0}$. It holds that $W^{k+1} - W^k = y + \Theta' \Lambda^k (W^{k+1} - W^k)$. Applying Lemma 3 implies that $W^{k+1} - W^k \geq y \geq \vec{0}$. This in turn implies that $\Lambda^{k+1} \geq \Lambda^k$. If $\Lambda^k = \Lambda^{k-1}$, it follows that $W^{k+1} = W^k$ and $\Lambda^k W^k = W^k \vee \vec{0}$. Hence, W^k is a solution to $W = u + \Theta(W \vee \vec{0})$. If $\Lambda^k \neq \Lambda^{k-1}$, the procedure has to be continued. The iteration has to stop after at most n steps because $\Lambda^k \leq \mathbf{I}$ for all k.

To prove that p^i is decreasing note that $p^1 \leq p^0 = \bar{p}$ by construction. Now suppose $p^0 \geq p^1 \geq \cdots \geq p^i$. $W^*(p)$ is increasing in p. Hence, $W^*(p^i) \leq W^*(p^{i-1})$ and therefore $p^{i+1} \leq p^i$. Now suppose the series converges to some $\tilde{p}$. This implies that

$$\tilde{p} = [(W^*(\tilde{p}) + \bar{p}) \vee \vec{0})] \wedge \bar{p} = \left\{ \left[e + \Pi' \tilde{p} + \Theta(W^*(\tilde{p}) \vee \vec{0}) \right] \vee \vec{0} \right\} \wedge \bar{p}.$$

[7] The fictitious default algorithm works if each p^i is a supersolution. This can be guaranteed for $e > \vec{0}$ and $p^0 = \bar{p}$. For $e \ngeq \vec{0}$ this property may not hold.

So $\tilde{p}$ is a clearing vector. Next note that $W^*(p^0) \geq W^*(p^+)$. This implies that $p^1 \geq p^+$. Now suppose it holds for i up to k that $p^i \geq p^+$. Hence, $W^*(p^k) \geq W^*(p^+)$. But this implies that $p^{k+1} \geq p^+$ and $\tilde{p} \geq p^+$. As p^+ is the largest clearing vector by assumption, $\tilde{p} = p^+$. □

The proposed solution algorithm detects the same sequence of defaults as the fictitious default algorithm. Banks defaulting in the first round are those that default even if their claims are honored fully. Banks defaulting in later rounds are dragged into default by their interbank counter parties.

5. Comparative Statics

Without cross holdings the clearing vector is a concave function of e and $\bar{p}$ as is shown by Eisenberg and Noe (2001). A simple example shows that the clearing vector is not concave as soon as cross holdings are included.

Example 3. *Assume the network is characterized by the following parameters:*

$$e = \begin{pmatrix} 0 \\ \lambda \\ -\frac{1}{10} \end{pmatrix}, \quad \Pi = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}, \quad \bar{p} = \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}, \quad \Theta = \begin{pmatrix} 0 & \frac{1}{2} & 0 \\ 0 & 0 & 0 \\ 0 & \frac{1}{4} & 0 \end{pmatrix}$$

Table 1 shows the clearing vectors and equity values as a function of $\lambda \in \mathbb{R}^n$. The clearing payments of banks 1 and 3 are not concave in λ. The equity values of banks 1 and 2 are not convex in λ.

In Figure 1 the equity value of bank 2, V_2^*, increases faster than exogenous income e_2, i.e. $\frac{\partial V_2^*}{\partial e_2} > 1$. This seems to suggest that injecting money into the network might increase the value of the bank by more than the supplied money. Let $p^*(e)$ and $V^*(p^*(e), e)$ be the clearing vector and the equity values corresponding to the financial network consisting of $(e, \Pi, \bar{p}, \Theta)$. Suppose some outside investor injects money into the network such that $\tilde{e} \geq e$. Using that $W^*(p, e)$ increases in e and p it follows that $p^*(\tilde{e}) \geq p^*(e)$ and $V^*(p^*(\tilde{e}), \tilde{e}) \geq V^*(p^*(e), e)$. As both $p^*(\tilde{e})$ and $p^*(e)$ are clearing vectors, it holds that

$$\tilde{e} + (\Pi' - \mathbf{I})p^*(\tilde{e}) + \Theta' V^*(p^*(\tilde{e}), \tilde{e}) \geq e + (\Pi' - \mathbf{I})p^*(e) + \Theta' V^*(p^*(e), e).$$

Putting this together and premultiplying by $\vec{1}'$ yields

$$\vec{1}'(\mathbf{I} - \Theta')(V^*(p^*(\tilde{e}), \tilde{e}) - V^*(p^*(e), e)) + \vec{1}'(\mathbf{I} - \Pi')(p^*(\tilde{e}) - p^*(e)) \leq \vec{1}'(\tilde{e} - e).$$

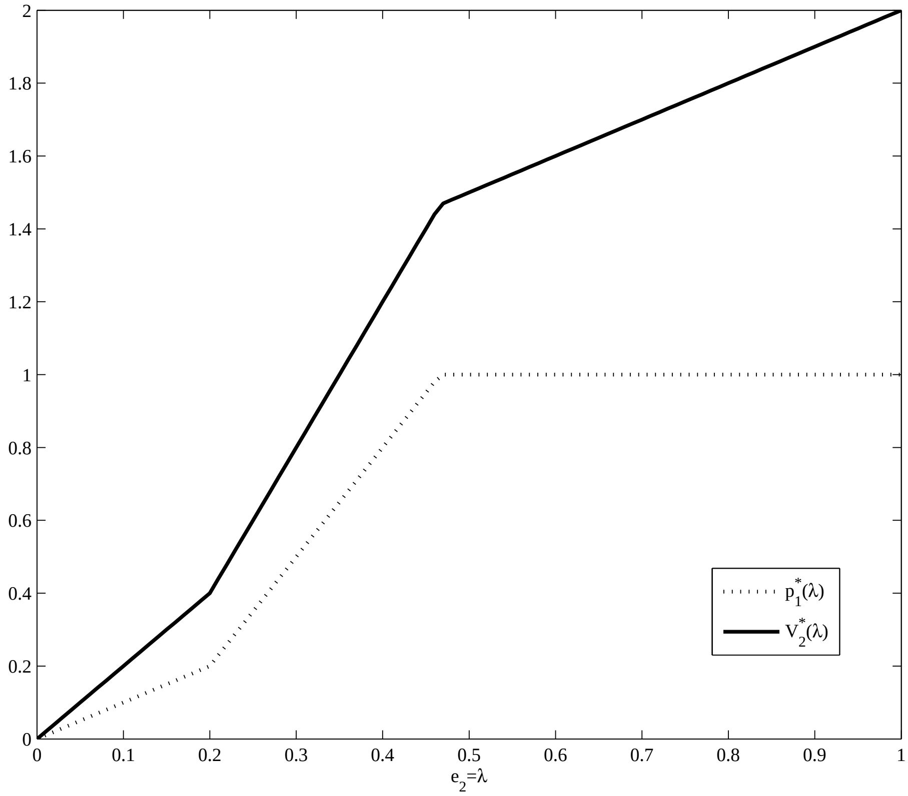

Figure 1: Debt payments of node 1 (dotted line) and the equity value of node 2 (solid line) as functions of $e_2 = \lambda$ in Example 3. Debt payments are not concave and the equity value is not convex in λ.

$\lambda \le 0$	$p^* = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$	$V^* = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$
$\lambda \in [0, 0.2]$	$p^* = \begin{pmatrix} \lambda \\ 0 \\ 0 \end{pmatrix}$	$V^* = \begin{pmatrix} 0 \\ 2\lambda \\ 0 \end{pmatrix}$
$\lambda \in [0.2, \frac{7}{15}]$	$p^* = \begin{pmatrix} 3\lambda - \frac{2}{5} \\ 0 \\ \lambda - \frac{1}{5} \end{pmatrix}$	$V^* = \begin{pmatrix} 0 \\ 4\lambda - \frac{2}{5} \\ 0 \end{pmatrix}$
$\lambda \in [\frac{7}{15}, \frac{17}{5}]$	$p^* = \begin{pmatrix} 1 \\ 0 \\ \frac{\lambda}{4} + \frac{3}{20} \end{pmatrix}$	$V^* = \begin{pmatrix} \frac{3}{4}\lambda - \frac{7}{20} \\ \lambda + 1 \\ 0 \end{pmatrix}$
$\lambda \ge \frac{17}{5}$	$p^* = \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}$	$V^* = \begin{pmatrix} \frac{1}{2}(\lambda + 1) \\ \lambda + 1 \\ \frac{1}{4}(\lambda + 1) - \frac{11}{10} \end{pmatrix}$

Table 1: Clearing vector and equity value as functions of banks 2's income λ in Example 3.

By increasing e to $\tilde{e}$ the value of an outside investor's portfolio changes by

$$\epsilon'(V^*(p^*(\tilde{e}), \tilde{e}) - V^*(p^*(e), e)) + \delta'(p^*(\tilde{e}) - p^*(e)).$$

Even if the entire network is owned by a single outside investor, i.e. $\delta' = \vec{1}'(\mathbf{I} - \Pi')$ and $\epsilon' = \vec{1}'(\mathbf{I} - \Theta')$, the amount gained will never exceed the amount injected, i.e. $\vec{1}'(\tilde{e} - e)$. If there are no bankruptcy costs, it does never pay to bail out.

Theorem 7. *Given two financial networks $(e, \Pi, \bar{p}, \Theta)$ and $(\tilde{e}, \Pi, \bar{p}, \Theta)$ with $\tilde{e} \ge e$ then it holds that*

$$\vec{1}'(\mathbf{I} - \Theta')(V^*(p^*(\tilde{e}), \tilde{e}) - V^*(p^*(e), e)) + \vec{1}'(\mathbf{I} - \Pi')(p^*(\tilde{e}) - p^*(e)) \le \vec{1}'(\tilde{e} - e).$$

In particular, for $\tilde{e} \ge \vec{0}$ the value of debt and equity held by outside investors equals the sum of the exogenous income across the banks, i.e.

$$\vec{1}'(\mathbf{I} - \Theta')V^*(p^*(\tilde{e}), \tilde{e}) + \vec{1}'(\mathbf{I} - \Pi')p^*(\tilde{e}) = \vec{1}'\tilde{e}.$$

Proof. Only the second claim remains to be shown. Let $e = \vec{0}$. Evidently, $\vec{0}$ is a clearing vector for the network $(\vec{0}, \Pi, \bar{p}, \Theta)$ with equity values $V^*(\vec{0}, \vec{0}) = \vec{0}$. If there is another clearing vector $\hat{p}(\vec{0})$, Theorem 5 implies that $\vec{1}'(\mathbf{I}-\Pi')\hat{p}(\vec{0}) = 0$ and $\vec{1}'(\mathbf{I}-\Theta')V^*(\hat{p}(\vec{0}), \vec{0}) = \vec{0}$. The inequality turns into an equality as $V^*(p^*(\vec{0}), \vec{0})) = \vec{0}+\Pi' p^*(\vec{0})+\Theta' V^*(p^*(\vec{0}), \vec{0})) = \vec{0}$. $\square$

6. Seniority Structure

Eisenberg and Noe (2001) interpret e_i as exogenous operating cash flow. They restrict e_i to be nonnegative reasoning that any operating costs like wages can be captured by appending a "sink node" to the financial system. Such a sink node has no operating cash flow of its own, nor any obligations to other nodes. The implicit assumption is that the operating costs are of the same priority as the liabilities in the financial system. If these costs are of a higher priority, modeling them via a sink node is not correct.

Example 4. *Assume that the financial system consist of two banks. Bank* 1 *has an operating cash flow of* 0.5*. Bank* 2 *has revenues of* 2 *but has to pay wages of* 4*. In the interbank market bank* 1 *owes bank* 2 *one unit and vice versa. If wages have the same priority as the interbank liabilities, we append an additional node* 3 *to the system for the workers. So*

$$e = \begin{pmatrix} \frac{1}{2} \\ 2 \\ 0 \end{pmatrix}, \quad \Pi = \begin{pmatrix} 0 & 1 & 0 \\ \frac{1}{5} & 0 & \frac{4}{5} \\ 0 & 0 & 0 \end{pmatrix}, \quad \bar{p} = \begin{pmatrix} 1 \\ 5 \\ 0 \end{pmatrix}, \quad \Theta = \mathbf{0}_{3,3}.$$

Clearing the system yields

$$p^* = \begin{pmatrix} 1 \\ 3 \\ 0 \end{pmatrix} \quad \text{and} \quad \Pi' p^* + e - p^* = \begin{pmatrix} \frac{1}{10} \\ 0 \\ \frac{24}{10} \end{pmatrix}.$$

The shortfall of node 2 *is proportionally shared between bank* 1 *and the workers. Each of them loses* 40% *of the promised payments. If we assume by contrast that wages are of a higher priority, the sink node approach can not be used. Yet, the problem is still well defined and can be solved. The system*

$$e = \begin{pmatrix} \frac{1}{2} \\ -2 \end{pmatrix}, \quad \Pi = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \bar{p} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

has the solution

$$p^* = \begin{pmatrix} \frac{1}{2} \\ 0 \end{pmatrix}, \quad \Pi' p^* + e - p^* = \begin{pmatrix} 0 \\ -\frac{3}{2} \end{pmatrix}$$

In this case node 1 *is bankrupt. It loses all payments promised by node* 2*. Node* 2 *is not able to pay off its obligations to the workers either* ($e_2 + \Pi_{1,2} p_1^* < 0$)*. The workers lose* 1.5 *of the promised payments.*

Finally, suppose there is a simple bilateral netting agreement between banks 1 *and* 2 *which stipulates that crosswise nominal obligations are netted. In this case bank* 1 *has a value of* 1/2 *and the entire losses have to be borne by the workers. In this case interbank obligations would be of the highest priority.*

The example demonstrates that the introduction of sink nodes is not as innocuous as it may seem. If there are different levels of priority, the amount available to pay off the most junior debt might be negative. Bank i's income e_i in the previous sections can be interpreted as the amount available to pay off the most junior debt. This makes it necessary not to restrict e_i to be nonnegative.

To adapt the framework to a more elaborate seniority structure I introduce seniority classes. Different liabilities are in the same seniority class if – in case of default – repayment is rationed proportionally between them.[8] Each bank may have a different number of priority classes S_i. Let S^* be the maximum of these S_i. Assume that debt claims in class 1 are of the highest priority, i.e. have to be satisfied first, then the claims in class 2 sequentially up to class S^* are satisfied. Debt claims include interbank positions as well as obligations to parties outside the banking system such as depositors or bondholders. Denote by $\bar{p}_{is} = \sum_{j=1}^{N} L_{ijs} + D_{is}$ the liabilities of bank i in class s. Define

$$\Pi_{ijs} = \begin{cases} \frac{L_{ijs}}{\bar{p}_{is}} & \text{if } \bar{p}_{is} > 0 \\ 0 & \text{otherwise} \end{cases}$$

and assume that if bank i has no debt in seniority class s then it has no debt in seniority class $s+1$ either ($\bar{p}_{is} = 0$ implies $\bar{p}_{is+1} = 0$). Let $p_{\cdot s} = (p_{1s}, \cdots, p_{ns})'$ and

$$\Pi_s = \begin{pmatrix} \Pi_{11s} & \cdots & \Pi_{1ns} \\ \vdots & \ddots & \vdots \\ \Pi_{n1s} & \cdots & \Pi_{nns} \end{pmatrix}.$$

[8] Lando (2004, pp. 247) and Elsinger et al. (2006b) discuss the consequences of bilateral netting agreements in a network model. It is important to highlight that netting agreements can appropriately be taken into account only if different priority levels exist. As can be seen in Example 4 netting of nominal obligations is equivalent to assuming that the involved liabilities are of the highest priority.

In analogy to the case of just one seniority class equity values $V^*(p)$ for a given $p = (p_{11} \dots p_{1S^*}, p_{21} \dots p_{2S^*}, \dots, p_{n1} \dots p_{nS^*})$ are defined as a fixed point of $\Psi^1(\cdot; p, e, \Pi, \Theta) : \mathbb{R}^n_+ \to \mathbb{R}^n_+$

$$V^*(p) = [e + \sum_{s=1}^{S^*} (\Pi_s)' p_{\cdot s} - \sum_{s=1}^{S^*} p_{\cdot s} + \Theta' V^*(p)] \vee \vec{0}.$$

Lemma 4 guarantees that $V^*(p)$ exists and is unique given that Θ is a holding matrix. A clearing payment vector has to satisfy limited liability and the seniority structure of the liabilities including absolute priority of debt.

Definition 3. *$p^* \geq \vec{0}$ is a clearing vector if $\forall i \in \{1, \dots, n\}$ and $\forall T \in \{1, \dots, S^*\}$*

$$p^*_{iT} = \min \left(\max \left(e_i + \sum_{j=1}^{N} \sum_{s=1}^{S^*} \Pi_{jis} p^*_{js} - \sum_{s=1}^{T-1} p^*_{is} + \sum_{j=1}^{N} \Theta_{ji} V^*_j \left(p^* \right), 0 \right), \bar{p}_{iT} \right).$$

A clearing vector has the property that if debt in seniority class T is not fully honored ($p^*_{iT} < \bar{p}_{iT}$), debt in seniority class $T+1$ is not served at all ($p^*_{iT+1} = 0$). On the other hand repaying at least a fraction of the debt in seniority class $T+1$, i.e. $p^*_{iT+1} > 0$, implies that debt in seniority class T is repaid in full. The definition insures that debt is paid off according to seniority. As a consequence of a detailed priority structure e_i might as well be assumed to be nonnegative.

The introduction of a detailed seniority structure does not change the main results which are proved in the Appendix.

Theorem 8. *Provided that Θ is a holding matrix it holds that*

1. *there exist a greatest and a least clearing vector,*
2. *the wealth of outside investors does not depend on the chosen clearing vector, and*
3. *if there is no subset of banks $\mathcal{I}$ such that for each bank in $\mathcal{I}$ either $\sum_{j \in \mathcal{I}} \Theta_{ij} = 1$ or $\sum_{j \in \mathcal{I}} \Pi_{ijs} = 1$ for some s then the clearing vector is unique.*

There are two ways to calculate a clearing vector. It can be done directly by a slight modification of the procedure described in Theorem 6. Start with $p^0 = \bar{p}$ and calculate $W^*(p^0)$ using

$$W^*(p) = e + \sum_{s=1}^{S^*} (\Pi_s)' p_{\cdot s} - \sum_{s=1}^{S^*} \bar{p}_{\cdot s} + \Theta(W^*(p) \vee \vec{0}).$$

Let

$$p^k_{iT} = \left\{ \left[W^*_i(p^{k-1}) + \sum_{s=T}^{S^*} \bar{p}_{is} \right] \vee 0 \right\} \wedge \bar{p}_{iT}$$

and iterate the procedure. $W^*(p)$ is increasing in p implying that $p^k \leq p^{k-1}$ for all k and $lim_{k\to\infty} p^k = p^+$ where p^+ denotes the largest clearing vector.

An alternative approach utilizes a sequential clearing procedure in which first a clearing vector for the most *junior* liabilities is calculated assuming all other claims are fully honored. If a bank is not able to honor any of its junior liabilities, we reduce the payments for the liabilities next to the most junior and so on. To formalize this, let $H = (H_1, \ldots, H_n)$ be a n-tuple of seniority classes and define

$$e_{iH} = e_i + \sum_{j=1}^{n} \sum_{s=1}^{H_j - 1} \Pi_{jis} \bar{p}_{js} - \sum_{s=1}^{H_i - 1} \bar{p}_{is}$$

$$\Pi_H = \begin{pmatrix} \Pi_{11H_1} & \cdots & \Pi_{1nH_1} \\ \vdots & \ddots & \vdots \\ \Pi_{n1H_n} & \cdots & \Pi_{nnH_n} \end{pmatrix}, \quad p_H = \begin{pmatrix} p_{1H_1} \\ \vdots \\ p_{nH_n} \end{pmatrix}, \quad \bar{p}_H = \begin{pmatrix} \bar{p}_{1H_1} \\ \vdots \\ \bar{p}_{nH_n} \end{pmatrix}.$$

e_{iH} is exogenous income of bank i plus all claims against other banks j that are more senior than seniority class H_j minus all liabilities of bank i that are more senior than H_i. Note that $e_H = (e_{1H}, \ldots, e_{nH})'$ is not necessarily positive even if e is positive. $\bar{p}_{iH_i}$ is the nominal obligation of bank i in seniority class H_i. Π_{ijH_i} is the fraction of nominal liabilities of bank i in seniority class H_i that it owes bank j.

To calculate a clearing vector start with $H^0 = (S_1, \ldots, S_n)$ where S_i is the most junior debt of bank i. Calculate a clearing vector $\hat{p}_{H^0}$ for the system $(e_{H^0}, \Pi_{H^0}, \bar{p}_{H^0}, \Theta)$ by the procedure described in Theorem 6. If $e_{H^0} + \Pi'_{H^0}\hat{p}_{H^0} - \hat{p}_{H^0} + \Theta' V(p^*_{H^0}) \geq \vec{0}$, we are done. Otherwise, let $\Lambda^0 = diag(e_{H^0} + \Pi'_{H^0}\hat{p}_{H^0} - \hat{p}_{H^0} + \Theta' V(\hat{p}_{H^0}) < \vec{0})$. If $\Lambda^0_{ii} = 1$, bank i is not only in default. It is not able to repay even *any* debt in seniority class H^0_i, $\hat{p}_{iH^0_i} = 0$. The amount available to cover liabilities in H^0_i is actually negative. For a bank i in Λ^0 we discard debt in seniority class H^0_i and set $H^1 = H^0 - \mathbf{1}'\Lambda^0$.

If we apply the clearing procedure from Theorem 6 to the system $(e_{H^1}, \Pi_{H^1}, \bar{p}_{H^1}, \Theta)$, it could happen that the equity value of a bank i in Λ^0 turns positive as debt in class H^0_i was deleted. To see why this problem does not occur note that

$$e_{H^0} + (\Pi'_{H^0} - \mathbf{I})\hat{p}_{H^0} = e_{H^1} + (\Pi'_{H^1} - \mathbf{I}) \left[(\mathbf{I} - \Lambda^0)\hat{p}_{H^0} + \Lambda^0 \bar{p}_{H^1} \right].$$

It is easy to verify that $\tilde{p} = (\mathbf{I} - \Lambda^0)\hat{p}_{H^0} + \Lambda^0 \bar{p}_{H^1} \geq \Phi^1(\tilde{p}; \Pi_{H^1}, \bar{p}_{H^1}, e_{H_1}, \Theta)$ and $\tilde{p}_i = \bar{p}_{iH^1} > \Phi^1_i(\tilde{p}; \Pi_{H^1}, \bar{p}_{H^1}, e_{H_1}, \Theta)$ for $i \in \Lambda^0$. $\tilde{p}$ is a supersolution but not a clearing vector. The set of supersolutions consists of all clearing vectors and the set of all p that are greater or equal to the largest clearing vector. This implies that the clearing vector $\hat{p}_{H^1}$ of $(e_{H^1}, \Pi_{H^1}, \bar{p}_{H^1}, \Theta)$ is smaller or equal to $\tilde{p}_{H^1} \leq \bar{p}_{H^1}$ and $\hat{p}_{iH^1} < \bar{p}_{iH^1}$ for all $i \in \Lambda^0$.

The value of a bank i that is not able to repay any debt in seniority class H_i^0 is zero.

Let $\Lambda^k = diag(e_{H^k} + \Pi'_{H^k}\hat{p}_{H^k} - \hat{p}_{H^k} + \Theta' V(\hat{p}_{H^k}) < \vec{0})$ and $H^{k+1} = H^k - \mathbf{1}'\Lambda^k$ where $\hat{p}_{H^k}$ is the clearing vector for the system $(e_{H^k}, \Pi_{H^k}, \bar{p}_{H^k}, \Theta)$. Clear the system $(e_{H^{k+1}}, \Pi_{H^{k+1}}, \bar{p}_{H^{k+1}}, \Theta)$ to get $\hat{p}_{H^{k+1}}$. Iterate this procedure until $\Lambda^k = \mathbf{0}_{n,n}$. Under the assumption that e is nonnegative the algorithm terminates after finitely many steps.

Example 5. *Using different priority classes we may rewrite Example 3 to show that even if $e > 0$, p^* is not necessarily concave in e. To do this interpret e in Example 3 as the net position after subtracting high priority debt from income. Assume that the counterparties of the highest priority debt are not part of the network. These liabilities are given by $D_{\cdot 1} = (D_{11}, D_{21}, D_{31})' = \left(1, 1, \frac{11}{10}\right)'$. There are no liabilities of the same priority within the network and hence $L_1 = \mathbf{0}_{3,3}$. Let $e = (1, 1+\lambda, 1)'$. So the net position after clearing highest seniority debt is equal to e in Example 3. The other parameters are given by $D_{\cdot 2} = \vec{0}$,*

$$L_2 = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}, \textit{ and } \quad \Theta = \begin{pmatrix} 0 & \frac{1}{2} & 0 \\ 0 & 0 & 0 \\ 0 & \frac{1}{4} & 0 \end{pmatrix}.$$

Hence, $\bar{p} = \left(1, 1, \frac{11}{10}, 1, 0, 1\right)$. It is easy to verify that the clearing payments of node 1 in seniority class 2 equal the clearing payment of node 1 in Example 3. Hence, p^* is not concave in e.

7. Conclusions

In this paper I analyze networks of financial institutions that are linked with each other via debt and equity claims. Limited liability of equity and a detailed seniority structure of debt are taken into account explicitly. The values of these claims are finite but not necessarily unique. Yet, whenever an outside investor holds a claim against a bank the value of this claim is unique. By adjusting the fictitious default algorithm developed in Eisenberg and Noe (2001) debt and equity values can be determined.

As long as bankruptcy costs are zero it never pays to bail out insolvent banks. Introducing bankruptcy costs does not change the main result that a solution to the clearing problem exists. Yet, bailing out insolvent banks or forgiving debt may become profitable.

The model is static. But by the inclusion of a detailed seniority structure it allows to take the timing of the payments into account and has thereby a dynamic flavor. The contagion effects of a negative shock to the economy (low realization of e) can be analyzed more precisely than in the case were all liabilities are modeled as pari passu.

The model presented is part of a simulation software at the Oesterreichische Nationalbank to assess the stability of the Austrian banking sector.[9] The simulations rest on the assumption that exogenous income e is a multidimensional random variable. For each draw of e the system is cleared. If e is drawn from the objective distribution, default probabilities and contagion effects can be assessed. If e is drawn from the risk neutral measure, the value of debt and equity can be determined by averaging across the simulations.

[9] A detailed description of the software is given in a technical document (Boss et al. (2006)) which is available upon request.

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A. Appendix

Lemma 1. *Let $\Theta \in [0,1]^{n\times n}$ be a matrix of interbank share holdings and let $\mathbf{I}$ be the $n \times n$ identity matrix. $(\mathbf{I} - \Theta')$ is invertible if and only if Assumption 1 is satisfied, i.e. Θ is a holding matrix.*

Proof. As $\mathbf{I} - \Theta' = (\mathbf{I} - \Theta)'$, it suffices to show that $(\mathbf{I} - \Theta)$ is invertible. Assume that there is a subset $\mathcal{I} \subset \{1, \ldots, n\}$ such that $\sum_{j\in\mathcal{I}} \Theta_{ij} = 1$ for all $i \in \mathcal{I}$. Let x be an $n \times 1$ vector with components $x_i = 1$ if $i \in \mathcal{I}$ and $x_i = 0$ otherwise. Clearly, $(\mathbf{I} - \Theta)x = \vec{0}$ where $\vec{0}$ denotes the $n \times 1$ dimensional zero vector. Thus $(\mathbf{I} - \Theta)$ is not invertible.

Now assume that $(\mathbf{I} - \Theta)$ is not invertible. Then there exists a vector $x \neq \vec{0}$ such that $(\mathbf{I} - \Theta)x = \vec{0}$. Writing down this system equation by equation we have a linear system given by

$$x_i = \sum_{j=1}^{n} \Theta_{ij} x_j \quad \text{for} \quad i = 1, ..., n.$$

Taking absolute values on both sides and applying the triangle inequality yields

$$|x_i| = |\sum_{j=1}^{n} \Theta_{ij} x_j| \leq \sum_{j=1}^{n} \Theta_{ij} |x_j| \quad \text{for} \quad i = 1, ..., n.$$

Now construct an index set $\mathcal{I} \subset \{1, \ldots, n\}$ as follows. The index i is in $\mathcal{I}$ if and only if $|x_i| \geq |x_j|$ for $j = 1, ..., n$. Since the triangle inequality holds for all i it holds in particular for all $i \in \mathcal{I}$. Thus we have

$$|x_i| \leq \sum_{j=1}^{n} \Theta_{ij} |x_j| \leq |x_i| \left(\sum_{j\in\mathcal{I}} \Theta_{ij} + \sum_{j\notin\mathcal{I}} \Theta_{ij} \right) \leq |x_i| \quad \text{for all} \quad i \in \mathcal{I}$$

with equality only if $\sum_{j\in\mathcal{I}} \Theta_{ij} = 1$. Hence, if $(\mathbf{I} - \Theta)$ is not invertible it has to hold that $\sum_{j\in\mathcal{I}} \Theta_{ij} = 1$ for all $i \in \mathcal{I}$. This violates Assumption 1. $\square$

Lemma 2. *Let Θ be an $n \times n$ holding matrix, let u be a $n \times 1$ vector, and let $\Lambda = diag(u > \vec{0})$. If $\Lambda \neq \mathbf{0}_{n,n}$ where $\mathbf{0}_{n,n}$ is an $n \times n$ matrix of zeros, it holds that $\vec{1}'\Lambda(\mathbf{I} - \Theta')\Lambda u > 0$.*

Proof. Λ is idempotent. Hence,

$$\vec{1}'\Lambda(\mathbf{I} - \Theta')\Lambda u = \vec{1}'\Lambda(\mathbf{I} - \Theta')\Lambda\Lambda u$$

$\Lambda u \geq \vec{0}$ by construction. $\vec{1}'\Lambda(\mathbf{I} - \Theta')\Lambda \geq \vec{0}'$ as no row sum of Θ exceeds one. This implies that $\vec{1}'\Lambda(\mathbf{I} - \Theta')\Lambda u \geq 0$. Now, suppose $\vec{1}'\Lambda(\mathbf{I} - \Theta')\Lambda u = 0$ and define the index set $\mathcal{I} := \{i | u_i > 0\}$. $\Lambda \neq \mathbf{0}_{n,n}$ implies that $\mathcal{I}$ is not empty. It has to hold that

$$0 = \sum_{i \in \mathcal{I}} u_i - \sum_{i \in \mathcal{I}} \sum_{j \in \mathcal{I}} \Theta_{ji} u_j = \sum_{i \in \mathcal{I}} u_i - \sum_{j \in \mathcal{I}} \left(\sum_{i \in \mathcal{I}} \Theta_{ji} \right) u_j$$

This implies that $\sum_{i \in \mathcal{I}} \Theta_{ji} = 1$ for all $j \in \mathcal{I}$. But this violates Assumption 1. □

Lemma 3. *Let Θ be a $n \times n$ holding matrix and let $y \geq \vec{0}$ be a $n \times 1$ vector. Then there exists a unique $x \geq y$ such that $x = y + \Theta' x$.*

Proof. Lemma 1 implies that x is unique. Now, suppose $x \ngeq y$ and let $\Lambda = diag(x < y)$.

$$\Lambda x = \Lambda y + \Lambda \Theta' \Lambda x + \Lambda \Theta' (\mathbf{I} - \Lambda) x$$

By construction $(\mathbf{I} - \Lambda) x \geq (\mathbf{I} - \Lambda) y$. So the last equation may be rewritten as

$$\Lambda (\mathbf{I} - \Theta') \Lambda (x - y) \geq \Lambda \Theta' y.$$

Premultiplying by $\vec{1}'$ yields

$$\vec{1}' \Lambda (\mathbf{I} - \Theta') \Lambda (x - y) \geq \vec{1}' \Lambda \Theta' y \geq 0.$$

If $\Lambda \neq \mathbf{0}_{n,n}$ the left hand side is smaller than 0 by Lemma 2. Therefore, $x \geq y$. □

Lemma 4. *Let $u \in \mathbb{R}^n$ and Θ be a holding matrix. Then the map $F(\cdot; u) : \mathbb{R}^n \to \mathbb{R}^n_+$*

$$F(V; u) = [u + \Theta' V] \vee \vec{0} \tag{14}$$

has a unique fixed point, $V^* \geq \vec{0}$.

Proof. Define $\tilde{V}$ by $\tilde{V} = [u \vee \vec{0}] + \Theta' \tilde{V}$. Given that Θ is a holding matrix , Lemma 1 implies that $\tilde{V}$ is well defined and unique. Lemma 3 implies $\tilde{V} \geq [u \vee \vec{0}] \geq \vec{0}$. Moreover,

$$F(\tilde{V}; u) = [u + \Theta' \tilde{V}] \vee \vec{0} = [u - (u \vee \vec{0}) + \tilde{V}] \vee \vec{0} \leq \tilde{V}.$$

As $F(V; u)$ is increasing on the complete lattice $[\vec{0}, \tilde{V}]$ the Tarski fixed point theorem (Theorem 11.E in Zeidler (1986)) implies that there exists a greatest and a least fixed point, V^+ and V^-, in the interval $[\vec{0}, \tilde{V}]$. Suppose V^*, not necessarily in $[\vec{0}, \tilde{V}]$, is yet another fixed point. Let $\Lambda = diag(V^* > V^-)$. Note that $\Lambda V^* = \Lambda(u + \Theta' V^*)$ and $\Lambda V^- \geq \Lambda(u + \Theta' V^-)$. This implies that $\Lambda(V^* - V^-) \leq \Lambda \Theta' \left(\Lambda(V^* - V^-) + (\mathbf{I} - \Lambda)(V^* - V^-) \right)$.

Rearranging and premultiplying by $\vec{1}'$ yields

$$\vec{1}'\Lambda(\mathbf{I}-\Theta')\Lambda(V^*-V^-) \leq \vec{1}'\Lambda\Theta'(\mathbf{I}-\Lambda)(V^*-V^-).$$

The right hand side of the above inequality is less than or equal to 0. Lemma 2 implies that the left hand side is larger than 0 as long as $\Lambda \neq \mathbf{0}_{n,n}$. So it has to hold that $V^* \leq V^-$. Evidently, $V^* \geq \vec{0}$. But as V^- is the smallest fixed point in $[\vec{0}, \tilde{V}]$ it follows that $V^* = V^-$ and the fixed point is unique. □

Lemma 5. *Let Θ be a holding matrix. Then*

$$W = u + \Theta'(W \vee \vec{0}) \tag{15}$$

has a unique solution W^* for any $n \times 1$ vector u. If u^1 and u^2 are two $n \times 1$ vectors such that $u^2 \geq u^1$ then $W^*(u^2) - W^*(u^1) \geq u^2 - u^1$ and $(\mathbf{I}-\Theta)(W^*(u^2) - W^*(u^1)) \leq u^2 - u^1$ where $W^*(u^1)$ and $W^*(u^2)$ are the respective fixed points.

Proof. Lemma 4 establishes that $V^* = [u + \Theta' V^*] \vee \vec{0}$ is unique. Let $X = u + \Theta' V^*$. It is easy to see that X solves (15). On the other hand if W^* solves (15) then $X = [W^* \vee \vec{0}]$ is a fixed point of $F(\cdot; u)$. To prove uniqueness assume there exist two solutions, W^1 and W^2. As $F(\cdot; u)$ has a unique fixed point, $[W^1 \vee \vec{0}] = [W^2 \vee \vec{0}]$. But this in turn implies that $W^1 = W^2$. Hence, Equation (15) has a unique solution.

To prove the second claim assume that u^1 and u^2 are two vectors in $\mathbb{R}^n$ such that $u^2 \geq u^1$. Let $W^*(u^1) = u^1 + \Theta'(W^*(u^1) \vee \vec{0})$ and $W^*(u^2) = u^2 + \Theta'(W^*(u^2) \vee \vec{0})$ be the respective fixed points. Let $x = u^2 - u^1 \geq \vec{0}$. It holds that

$$W^*(u^2) - W^*(u^1) = x + \Theta'([W^*(u^2) \vee \vec{0}] - [W^*(u^1) \vee \vec{0}])$$

Let $\Lambda = diag(W^*(u^1) > W^*(u^2))$. Note that $\Lambda([W^*(u^2) \vee \vec{0}] - [W^*(u^1) \vee \vec{0}]) \geq \Lambda(W^*(u^2) - W^*(u^1))$ and $(I - \Lambda)([W^*(u^2) \vee \vec{0}] - [W^*(u^1) \vee \vec{0}]) \geq \vec{0}$. Hence,

$$\Lambda(W^*(u^2) - W^*(u^1)) \geq \Lambda x + \Lambda\Theta\Lambda(W^*(u^2) - W^*(u^1))$$

Rearranging and premultiplying by $\vec{1}'$ yields

$$\vec{1}'\Lambda(\mathbf{I}-\Theta')\Lambda(W^*(u^2) - W^*(u^1)) \geq \vec{1}'\Lambda x$$

The right hand side is larger or equal to 0. The left hand side is smaller than 0 by

Lemma 2 unless $\Lambda = \mathbf{0}_{n,n}$. Hence, $W^*(u^2) \geq W^*(u^1)$. This implies

$$W^*(u^2) - W^*(u^1) \geq ([W^*(u^2) \vee \vec{0}] - [W^*(u^1) \vee \vec{0}]) \geq 0$$

and therefore $W^*(u^2) - W^*(u^1) \geq u^2 - u^1$ and $(\mathbf{I} - \Theta')(W^*(u^2) - W^*(u^1)) \leq u^2 - u^1$. $\square$

Theorem 8. Provided that Θ is a holding matrix it holds that

1. there exist a greatest and a least clearing vector,
2. the wealth of outside investors does not depend on the chosen clearing vector, and
3. if there is no subset of banks $\mathcal{I}$ such that for each bank in $\mathcal{I}$ either $\sum_{j \in \mathcal{I}} \Theta_{ij} = 1$ or $\sum_{j \in \mathcal{I}} \Pi_{ijs} = 1$ for some s then the clearing vector is unique.

Proof. In analogy to the case of only one seniority class a clearing vector can be defined as a fixed point of the map
$\Phi^1(p) = (\Phi^1_{11} \ldots \Phi^1_{1S^*}, \Phi^1_{21} \ldots \Phi^1_{2S^*}, \ldots, \Phi^1_{n1} \ldots \Phi^1_{nS^*}) : [\vec{0}, \bar{p}] \to [\vec{0}, \bar{p}]$ defined by

$$\Phi^1_{iT}(p) = \left\{ \left[e_i + \sum_{j=1}^{N} \sum_{s=1}^{S^*} \Pi_{jis} p_{js} - \sum_{s=1}^{T-1} p_{is} + \sum_{j=1}^{N} \Theta_{ji} V^*_j(p) \right] \vee 0 \right\} \wedge \bar{p}_{iT}. \tag{16}$$

Moreover,

$$W^*(p) = e + \sum_{s=1}^{S^*} (\Pi_s)' p_{\cdot s} - \sum_{s=1}^{S^*} \bar{p}_{\cdot s} + \Theta'(W^*(p) \vee \vec{0}) \tag{17}$$

and

$$\Phi^2_{iT}(p) = \left\{ \left[e_i + \sum_{j=1}^{N} \sum_{s=1}^{S^*} \Pi_{jis} p_{js} - \sum_{s=1}^{T-1} \bar{p}_{is} + \sum_{j=1}^{N} \Theta_{ji} (W^*_j(p) \vee 0) \right] \vee 0 \right\} \wedge \bar{p}_{iT}. \tag{18}$$

Part a) The existence of a greatest and a least clearing vector is proved in two steps. The first step is to verify that any solution of Φ^1 is a solution of Φ^2 and vice versa. The second step consists of showing that Φ^2 is increasing in p and applying the Tarski fixed point theorem.

I prove that any supersolution of Φ^1 is a supersolution of Φ^2 and vice versa. To show that any solution of Φ^1 is a solution of Φ^2 and vice versa is analogous. Suppose that $\hat{p}$ is a supersolution of Φ^1, i.e. $\hat{p} \geq \Phi^1(\hat{p})$. Let $X = e + \sum_{s=1}^{S^*} (\Pi_s)' \hat{p}_{\cdot s} - \sum_{s=1}^{S^*} \bar{p}_{\cdot s} + \Theta' V^*(\hat{p})$.

Evidently, $V^*(\hat{p}) \geq X$. Whenever, $V_i^*(\hat{p}) > 0$ it has to hold that

$$0 < e_i + \sum_{j=1}^{N} \sum_{s=1}^{S^*} \Pi_{jis} \hat{p}_{js} - \sum_{s=1}^{S^*} \hat{p}_{is} + \sum_{j=1}^{N} \Theta_{ji} V_j^*(p).$$

This implies that

$$\hat{p}_{iT} < e_i + \sum_{j=1}^{N} \sum_{s=1}^{S^*} \Pi_{jis} \hat{p}_{js} - \sum_{s=1}^{T-1} \hat{p}_{is} + \sum_{j=1}^{N} \Theta_{ji} V_j^*(p) \quad \forall T \in \{1, \ldots, S^*\}.$$

Given that $\hat{p}$ is a supersolution of Φ^1 we get $\hat{p}_{iT} = \bar{p}_{iT}$ for all $T \in \{1, \ldots, S^*\}$. Therefore, $V^*(\hat{p}) = (X \vee \vec{0})$ and X solves (17). As $\Phi^1(\hat{p}) \geq \Phi^2(\hat{p})$ the vector $\hat{p}$ is a supersolution of Φ^2, too. Now, assume that $\hat{p}$ is a supersolution of Φ^2. Let $X = (W^*(\hat{p}) \vee \vec{0})$. If $W_i^*(\hat{p}) \geq 0$ then

$$\bar{p}_{iT} \leq e_i + \sum_{j=1}^{N} \sum_{s=1}^{S^*} \Pi_{jis} \hat{p}_{js} - \sum_{s=1}^{T-1} \bar{p}_{is} + \sum_{j=1}^{N} \Theta_{ji} (W_j^*(p) \vee 0)$$

implying that $\hat{p}_{iT} = \bar{p}_{iT}$ for all $T \in \{1, \ldots, S^*\}$. Hence, for all i with $W_i^*(\hat{p}) \geq 0$ we get

$$X_i = e_i + \sum_{j=1}^{N} \sum_{s=1}^{S^*} \Pi_{jis} \hat{p}_{js} - \sum_{s=1}^{S^*} \hat{p}_{is} + \sum_{j=1}^{N} \Theta_{ji} X_j.$$

Suppose $W_i^*(\hat{p}) < 0$ and let H_i be the highest index such that $\hat{p}_{iH_i} = \bar{p}_{iH_i}$. If $H_i = S^*$ then

$$0 > e_i + \sum_{j=1}^{N} \sum_{s=1}^{S^*} \Pi_{jis} \hat{p}_{js} - \sum_{s=1}^{S^*} \hat{p}_{is} + \sum_{j=1}^{N} \Theta_{ji} X_j.$$

For $H_i < S^*$ it has to hold that

$$\bar{p}_{iH_i+1} > \hat{p}_{iH_i+1} \geq e_i + \sum_{j=1}^{N} \sum_{s=1}^{S^*} \Pi_{jis} \hat{p}_{js} - \sum_{s=1}^{H_i} \hat{p}_{is} + \sum_{j=1}^{N} \Theta_{ji} X_j(p) \tag{19}$$

as $\hat{p}$ is a supersolution. Hence.

$$0 \geq e_i + \sum_{j=1}^{N} \sum_{s=1}^{S^*} \Pi_{jis} \hat{p}_{js} - \sum_{s=1}^{S^*} \hat{p}_{is} + \sum_{j=1}^{N} \Theta_{ji} X_j(p).$$

So we may write

$$X = \left(e + \sum_{s=1}^{S^*} (\Pi_s)' \hat{p}_{\cdot s} - \sum_{s=1}^{S^*} \hat{p}_{\cdot s} + \Theta' X \right) \vee \vec{0}.$$

To prove that $\hat{p}$ is a supersolution of Φ^1 note that for all $s \leq H_i + 1$, $\Phi^1_{is} = \Phi^2_{is}$ and for all $s > H_i + 1$, it has to hold that $0 \geq \Phi^1_{is} \geq \Phi^2_{is}$ as can be seen by (19). The vector $\hat{p}$ is indeed a supersolution of Φ^1.

Φ^1 and Φ^2 have the same (super)solutions. Lemma 5 establishes that $W^*(p)$ and consequently $\Phi^2(p)$ are increasing in p. Applying the Tarski fixed point theorem yields that there exist a greatest and a least clearing vector for the problem.

Parts b) and c) Let p^1 and p^2 be two clearing vectors such that $p^2 \geq p^1$. Denote the corresponding equity values by V^1 and V^2. Without loss of generality it can be assumed that $e \geq 0$. This implies that for any clearing vector p^* the corresponding equity values can be written as

$$V^*(p^*) = e + \sum_{s=1}^{S^*} (\Pi_s)' p^*_{\cdot s} - \sum_{s=1}^{S^*} p^*_{\cdot s} + \Theta' V^*(p^*).$$

Let $\Lambda^0 = diag(V^2 > V^1)$ and $\Lambda^s = diag(p^2_{\cdot s} > p^1_{\cdot s})$ for each $s \in \{1, \ldots, S^*\}$. Let Λ characterize all banks that either belong to Λ^0 or some Λ^s, i.e. $\Lambda = \mathbf{I} - \prod_{s=0}^{S^*} (\mathbf{I} - \Lambda^s)$. Subtracting V^1 from V^2 and multiplying by $\vec{1}'\Lambda$ yields

$$\vec{1}'\Lambda(\mathbf{I} - \Theta')\Lambda^0(V^2 - V^1) = \sum_{s=1}^{S^*} \vec{1}'\Lambda(\Pi'_s - \mathbf{I})\Lambda^s(p^2_{\cdot s} - p^1_{\cdot s}).$$

For this equality to hold the left hand side and each summand on the right hand side has to equal 0.

$$\vec{1}'\Lambda(\mathbf{I} - \Theta')\Lambda^0 = \vec{1}'\Lambda(\Pi'_s - \mathbf{I})\Lambda^s = 0 \quad \text{for all } s.$$

Any bank i with a non–unique equity value has to be owned entirely by banks in Λ. An outside investor can not own any share of such a bank. If the clearing payment of a bank in seniority class s is non–unique then the bank owes all liabilities in this class to banks in Λ. No claims are held by outside investors. The value of the outside investor's portfolio is independent of the chosen clearing vector. If there is no subset of banks such that each bank is either owned entirely by banks in this subset or all obligees in at least one seniority class belong to this subset then the equity values and clearing payments have to be unique. □

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