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Actuarial Deductions for Early Retirement

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Actuarial Deductions for Early Retirement

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Abstract

The paper studies how the rates of deduction for early retirement have to be determined in PAYG systems in order to keep their budget stable. I show that the budget-neutral deductions depend on the specific rules of the pension system and on the choice of the discount rate which itself depends on the collective retirement behavior. For the commonly used fiction of a single individual deviating from the target retirement age the right choice is the market interest rate while for the alternative scenario of a stationary retirement distribution it is the internal rate of return of the PAYG system. In this case the necessary budget-neutral deductions are lower than under the standard approach used in the related literature. This is also true for retirement ages that fluctuate randomly around a stationary distribution. Long-run shifts (e.g. increases in the average retirement age) might cause problems for the pension system but these have to be dealt with by the general pension formulas not by the deduction rates.

Keywords: Pension System; Demographic Change; Financial Stability;
JEL-Classification: H55; J1; J18; J26

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Non-Technical Summary

It is often argued that individuals should have a high degree of flexibility when choosing their own retirement age. A crucial qualification to this statement, however, is that these individual retirement decisions should be "budget-neutral", i.e. they should leave the long-run budget of the pension system unaffected. This paper studies how the budget-neutral deductions for early retirement (and the budget-neural supplements for late retirement) should be determined in order to meet this goal.

The paper shows that the level of these budget-neutral deduction rates depends on two crucial factors: on the basic formulas of the pension system and on the assumption about collective retirement behavior. As far as the pension formulas are concerned it makes a difference whether the pension formula is independent of the retirement age (as is, e.g., the case for a pure defined benefit system) or whether it reacts rather strongly (as in a notional defined contribution (NDC) system). The collective retirement behavior, on the other hand, influences the discount rate that is used to calculate the present values of total pension benefits and of total contributions which are needed to derive the budget-neutral deduction rates.

In the paper it is shown that a NDC system will lead to a balanced budget without the need of additional deductions or supplements if the retirement age is stationary. In this case the costs of early retirement by some are exactly offset by the cost savings due to late retirement of others. In a second step it is demonstrated that in this stationary situation the appropriate discount rate is given by the internal rate of return (IRR) of the PAYG system. This result stands in contrast to the related literature that often argues that the use of the market interest rate is the necessary and almost self-evident choice to determine budget-neutral deduction rates.

Various extensions confirm the main result of the paper. First, it is shown that a pure NDC system without additional deductions is also able to guarantee an (approximately) balanced budget when the actual retirement distribution fluctuates randomly around a stable distribution. This constellation corresponds quite well to empirically observed patterns. Second, I demonstrate that the main result (market interest rates are unimportant for stationary retirement distributions) also holds for other PAYG systems (like defined benefit systems or accrual rate systems). Third, I show that the use of unnecessarily high deduction rates might also be compatible with a balanced budget but that it raises fairness concerns. In particular, such a policy implies excessive punishments for early retirement and excessive rewards for later retirement with probably unintended and undesired implications for the interpersonal distribution. Fourth, I discuss situations that involve long-run shifts in demographic variables that might pose a challenge for PAYG systems. I argue, however, that the appropriate reaction to these long-run changes have to be factored into the basic pension formulas rather than into the deduction rates.

1 Introduction

This paper discusses how pay-as-you-go (PAYG) pension systems have to determine actuarial deductions for early retirement and actuarial supplements for late retirement in order to remain financially balanced in the long run. Despite the fact that the levels of deductions and supplements are crucial parameters for pension design that are present in all real-world systems the existing literature on this issue is rather small and sometimes controversial. In this paper I use a simple model to discuss this topic in a systematic and comprehensive manner.

Deductions are necessary since an insured person who retires at an earlier age pays less contributions into the pension system than an otherwise identical individual and he or she also receives more installments of (monthly or annual) pension payments.[1] The deductions have to be determined in such a way that the net present value of these altered payment streams is zero. This calculation depends on two crucial factors. The first one is the definition of the pension formula. Many pension systems take the difference between the actual retirement age R and the target (or "statutory") retirement age R^* into account when assigning the pension payment. The stronger pension benefits react to the difference between the actual and the target age the weaker the need for additional deductions. In the paper I focus on three variants of PAYG systems: a defined benefit (DB) system that does not react to the actual retirement age R; an accrual rate (AR) system (as it is, e.g., in place in Germany or France) where the basic pension formula already implies a lower pension payment for earlier retirement; and a notional defined contribution (NDC) system (introduced in countries like Sweden, Italy or Poland) which is based on a formula that adjusts pension payments to the fact that early retirement is associated with fewer years of contributions *and* with more years of pension payments.

The second important factor to calculate budget-neutral deductions is the discount rate that is used to equalize the present value of costs and benefits. In the related literature the most common suggestion is to use the market interest rate. This is, e.g., argued by Werding (2007, p.21) on the grounds that the "government has to borrow the funds for premature pensions" and that the relevant interest rate for these transactions is "evidently the capital market interest rate", in particular the "risk-less interest rate for long-run government bonds". In the present paper I argue that this choice is not

[1] From now on I will focus on the case of early retirement and the associated deductions. All of the following statements and results, however, also hold for the opposite case of late retirement and associated supplements.

as self-evident as suggested in the related literature. In fact, the level of discount rates that is appropriate to calculate budget-neutral deduction rates depends on the *collective* retirement behavior. The standard approach of the literature uses a thought experiment in which everybody retires at the target retirement age R^* while only one individual retires at an earlier age. In this hypothetical scenario the behavior of the deviant individual causes an extra financing need and the market interest rate is in fact the appropriate concept for discounting the ensuing cash flows. Early retirement can, however, not only occur in the context of such a one-time-shock scenario. There exists, e.g., also an alternative scenario where people retire at different ages according to a retirement distribution that is stable over time. It is not clear from the outset whether the conventional wisdom also holds for this alternative assumption about collective behavior.

In this paper I look at this issue in detail. The main result of the analysis consists of two parts. First, I show that if the retirement ages follow a stationary distribution over time then a NDC system leads to a stable budget. The costs of early retirement are exactly offset by cost savings of late retirement and there is no financial need for additional deductions or supplements. Second, in this case the appropriate discount rate is given by the internal rate of return (IRR) of the PAYG system. Contrary to wide-spread claims in the literature, it is thus not necessary or "evident" to use the market interest rate for the determination of budget-neutral deductions. The rest of the paper adds elaborations, extensions and discussions of the central result.

First, I show that it is straightforward to amend the two other PAYG systems (DB and AR) in order to "mimic" the NDC system. It is thus sufficient to derive and discuss the main results for the NDC system since they can be easily extended to the other pension formulas. Second, for illustrative purposes I calculate deduction rates for realistic demographic scenarios. For a discount rate that equals the IRR of the PAYG system the deductions are between 5.5% and 7.0% (DB system) and 4.2% and 4.9% (AR system) while they are 0% for the NDC system. The use of higher discounts rates increases the annual deduction rates by slightly less than 1:1. In particular, for an interest rate that is 2% higher than the internal rate of return they range from 7.3% to 8.7% (for the DB system), 5.7% to 6.6% (for the AR system) and 1.8% to 1.9% (for the NDC system). Third, given the fact that different discount rates imply different deductions it is crucial to choose the appropriate assumption concerning collective retirement behavior. I discuss this issue and argue that there exist good reasons to regard the stationary retirement distribution as the more reasonable reference point. On the one hand, the one-time shock scenario cannot be extended over time. In the year after the single early retiree left the labor market the initial situation has changed and it is no longer appropriate to start with the thought experiment in which everybody retires at the target age. If every period a certain fraction of individuals retire early than one would ultimately end up with the other scenario of a stationary distribution. On the other hand, real-world retirement patterns do not correspond to such a degenerate distribution where almost everybody retires at the target age, but show a wider, rather stable and typically slowly changing distribution. Fourth, I look at various additional scenarios where the retirement age follows a non-stationary distribution. Also in many of these cases the appropriate deductions are below the value that are associated with the one-time-shock scenario. In particular, numerical simulations show that for situations where the actual retirement distributions fluctuate randomly around a stable distribution a pure NDC system is basically sufficient to guarantee a balanced budget and the additional deductions can be close to zero. Fifth, I demonstrate that even for a stationary distribution of retirement ages the choice of a higher discount rate might still be associated with a balanced budget if the target retirement age R^* is equal to the average actual retirement age. If this is not the case then the system will run permanent surpluses or deficits. The use of a higher discount rate is however not innocuous in both situations. In particular, it might look problematic from the viewpoint of the interpersonal distribution since it implies that early retirees have to pay deductions that are larger than what is necessary for budgetary stability while late retirees are offered larger-than-necessary supplements. Sixth, I briefly discuss various extensions of the model. In the first extension I generalize the basic model with rectangular mortality to a model with an arbitrary mortality structure. The main results continue to hold in this framework. Further extensions deal with additional heterogeneity along various dimensions and with pension systems that are unbalanced by construction. For the assumption of a stationary retirement distribution it is still the case in these extensions that the level of budget-neutral deductions is related to the IRR of the system and independent of the market interest rates. The deduction rates, however, might now have distributional consequences. Finally, I discuss situations that involve long-run shifts in demographic or economic variables that often pose a challenge for PAYG systems. An increase in the average retirement age, e.g., leads to a constellation where none of the common deduction rates (neither the one based on market interest rates nor the one based on the IRR) is able to implement a balanced budget. These long-run changes have to be reflected in the design of the basic pension formulas and they require more thorough considerations about intergenerational risk-sharing and redistribution. These difficult issues have to be treated separately from the more modest topic of how to determine budget-neutral deduction rates.

There exists a broad literature on actuarial adjustments from the perspective of the insured individuals. This is sometimes called the “microeconomic” or “incentive-compatible” viewpoint (cf. Börsch-Supan 2004, Queisser & Whitehouse 2006) since it focuses on the determination of deductions that leave an *individual* indifferent between retiring at the target or at an alternative age. This individual perspective has been used to study the impact of non-actuarial adjustments on early retirement (Stock & Wise 1990, Gruber & Wise 2000), the incentives to delay retirement (Coile et al. 2002, Shoven & Slavov 2014) and the simultaneous decisions on retirement, benefit claiming and retirement in structural life cycle models (Gustman & Steinmeier 2015). The present paper concentrates on the “macroeconomic viewpoint”, i.e. on actuarial adjustments from the perspective of the pension system.[2] The literature on this topic is smaller than the one on incentive-compatibility. Queisser & Whitehouse (2006) offer terminological discussions and they present evidence on the observed rates of adjustments. For a sample of 18 OECD countries they report an average annual deduction for early retirement of 5.1% and an average annual supplement of 6.2% for late retirement. They conclude that “most of the schemes analysed fall short of actuarial neutrality [and that] as a result they subsidize early retirement and penalize late retirement” (p.29). An intensive debate about this topic can be observed in Germany where the rather low annual deductions rates of 3.6% are frequently challenged. A number of researchers have supported higher rates of deductions based on market interest rates (Börsch-Supan & Schnabel 1998, Fenge & Pestieau 2005, Werding 2007, Brunner & Hoffmann 2010) while other participants have argued for keeping rates low stressing the lower IRR of the PAYG system (Ohsmann et al. 2003). Overviews of the debate can be found in Börsch-Supan (2004) and Gasche (2012).

The paper is organized as follows. In section 2 I present the basic deduction equation and I focus on a simple model that allows for analytical solutions. In section 3 I derive the level of budget-neutral deductions for various assumptions concerning collective retirement behavior. In section 4 I discuss the case of “excessive deductions”, i.e. of deductions that are higher than necessary for budgetary balance which might raise distributional concerns.

[2] “Actuarial neutrality” (either of the microeconomic or of the macroeconomic type) must also be distinguished from the notion of “actuarial fairness”. The latter concept is often used to describe a system in which the present value of expected contributions is ex-ante equal to the present value of expected pension payments (cf. Börsch-Supan 2004, Queisser & Whitehouse 2006). This is therefore a life-cycle concept while actuarial neutrality can be regarded as a “marginal concept” that focuses on the effect of postponing retirement by an additional year.

Section 5 and 6 discuss various extensions and section 7 concludes.

2 Simple framework

2.1 Set-up

I start with the benchmark approach of the related literature. In order to fix ideas I focus on the most simple case with a constant wage W and a stable demographic structure where all individuals start to work at age A, are continuously employed and die at age ω.[3]

There exists a PAYG pension system with a constant contribution rate τ, a target (or reference) retirement age R^* and a pension formula that determines the regular pension for each admissible retirement age R. In the most simple form the system only determines the pension level P^* that is promised for a retirement at the target age R^*. In this case the pension deductions are the only instrument to implement appropriate adjustments for early retirement. Many real-world pension systems, however, are based on a "formula pension" that depends on the target retirement age R^* *and* on the actual retirement age R thereby accounting (at least partially) for early retirement. This formula pension is denoted by $\widehat{P}(R, R^*)$ and I will discuss below various possibilities for its determination. For the moment, however, I leave it unspecified. Furthermore, for brevity I will often omit the arguments of $\widehat{P}(R, R^*)$ and other functions whenever there is no risk of ambiguity. Finally, the pension level at the target retirement age is given by $P^* \equiv \widehat{P}(R^*, R^*)$.

2.2 Deductions for a general discount rate

If an individual chooses to retire before the target age (i.e. $R < R^*$) then the actuarial neutral deduction for early retirement will reduce the formula pension payment $\widehat{P}$ in such a way that the retirement decision has no long-run effect on the budget of the social security system. In order to do so two effects have to be taken into account. First, for the periods between R and R^* the individual does not pay pension contributions and thus the system has a shortfall of revenues. Second, in these periods of early retirement the individual already receives pension payments and thus the system has to cover additional expenditures. The formula pension level $\widehat{P}$ thus has to be reduced by a factor X (that is

[3] In appendix C I will discuss the case where wages grow at rate $g(t)$ and where there exists mortality before the maximum age ω.

valid for the *entire* pension period) in order to counterbalance these two effects. The final pension will thus be given by $P = \widehat{P}X$. Using a continuous time framework the actuarial deduction factor X is implicitly defined as follows:

$$\int_R^{R^*} \left(\tau W + \widehat{P}X\right) e^{-\delta(a-R)}\,\mathrm{d}a = \int_{R^*}^{\omega} \left(P^* - \widehat{P}X\right) e^{-\delta(a-R)}\,\mathrm{d}a, \tag{1}$$

where δ is the social discount rate used to evaluate future payment streams. The left-hand side of equation (1) contains the twofold costs to the system due to early retirement (i.e. the period loss of contributions $\tau W$and the additional period expenditures $\widehat{P}X$). The right-hand side captures the benefits to the system since in the case of a retirement at the target age the pension without deductions would be P^* for all periods between R^* and ω which is now reduced to $P = \widehat{P}X$.[4]

The determination of the discount rate δ is a crucial issue. In fact, it will turn out that different approaches to calculate appropriate deductions differ primarily in their choice of the discount rate (see also Gasche 2012). If the costs of early retirement have to be financed by debt then the market interest rate seems to be the right choice, i.e. $\delta = r$. If, one the other hand, the budget of the system remains under control (e.g. because early retirement of some is counterbalanced by late retirement of others) then a lower interest rate like the internal rate of return of the PAYG system seems appropriate. I come back to this crucial issue later.

One can solve equation (1) for X which gives rise to a rather complicated expression. Linearization of this result (around $\delta = 0$) leads to the approximated value $\widetilde{X}$ given by:

$$\widetilde{X} = \frac{\omega - R^*}{\omega - R}\left[\frac{P^*}{\widehat{P}} + \frac{\tau W}{\widehat{P}}\frac{R - R^*}{\omega - R^*} + \frac{\delta}{2}(R - R^*)\left(\frac{P^*}{\widehat{P}} + \frac{\tau W}{\widehat{P}}\right)\right]. \tag{2}$$

[4]The same logic also holds for late retirement with $R > R^*$. In this case the equivalent to (1) is given by: $\int_{R^*}^{R}(\tau W + P^*)\, e^{-\delta(a-R)}\,\mathrm{d}a = \int_R^{\omega}\left(\widehat{P}X - P^*\right) e^{-\delta(a-R)}\,\mathrm{d}a$. This can be transformed to yield (1). Note that equation (1) could also be viewed as the condition that makes an individual indifferent between retirement at the age of R^* and retirement at an earlier age R. In this case the appropriate discount rate is given by his or her individual rate of time preference which is typically associated with the market interest rate. The deductions derived under this approach are sometimes termed "incentive compatible". They could also be called "actuarial neutral from the perspective of the insured person". A related concept is the "social security wealth" that is often used in this context to study the incentives for early or delayed retirement (see e.g. Stock & Wise 1990, Gruber & Wise 2000, Shoven & Slavov 2014). In the present I focus, however, on "budget neutral" deductions which could also be called "actuarial neutral from the perspective of the insuring system".

2.3 Different PAYG systems

In order to further evaluate expression (2) and to derive numerical values one has to specify how the formula pension level $\widehat{P}$ is determined. There exist various possibilities and I will discuss three variants that are often used in existing pension systems.

- **Defined Benefit (DB) System**: In this case there exists a target replacement rate q^* that is promised if an individual retires at the target retirement age R^*. In the generic DB case the pension formula is independent of the actual retirement age and does not reduce the target replacement rate, i.e. $\widehat{P}_{\mathrm{DB}}(R, R^*) = q^* W$.

- **Accrual Rate (AR) System**: Many countries have PAYG pensions systems in place that are somewhat more sensitive to actual retirement behavior than the DB system. In particular, in these systems the formula pension is reduced if retirement happens before the target age R^*. One popular example of such a system is built on the concept of an "accrual rate". For each period of work the individual earns an accrual rate κ^* that is specified in a way that the system promises the full replacement rate q^* only if the individual retires at the target retirement age $R = R^*$. This means that $\kappa^* = q^* \frac{1}{R^* - A}$ and $\widehat{P}_{\mathrm{AR}}(R, R^*) = \kappa^*(R - A)W = q^* \frac{R-A}{R^*-A} W$.[5]

- **Notional Defined Contribution (NDC) System**: This scheme has been established in Sweden and in a number of other countries and is increasingly popular. Its main principle is that at the moment of retirement at age R the total of contributions that an individual has accumulated over the working life $\tau W(R - A)$ is transformed into a period pension by dividing it by the remaining life expectancy $\omega - R$. This means that $\widehat{P}_{\mathrm{NDC}}(R, R^*) = \tau W \frac{R-A}{\omega - R}$.[6] Therefore in the NDC system the target retirement age R^* does not play a role and the formula pension just reacts to the actual retirement age R.

The pension for early (or late) retirement in each of the three cases $j \in \{\mathrm{DB}, \mathrm{AR}, \mathrm{NDC}\}$ is then given by $P_j = \widehat{P}_j X_j$ (where I skip again the function values). I call the ratio of the final pension P_j to the target pension P_j^* the total pension *reduction*. This reduction might be either due to stipulations of the formula pension $\widehat{P}_j$ or due to the influence of the explicit *deductions* X_j. For the DB system, e.g., the entire reduction follows from the

[5] A system like that is, e.g., in place in Austria. The earnings point system in Germany or France can also be directly related to this PAYG variant.

[6] Real-world NDC systems are more complicated due to non-stationary economic and demographic patterns. This is discussed in appendix C.

Table 1: Three simple PAYG systems

	(1)	(2)	(3)	(4)	(5)
Type (j)	$\widehat{P}_j$	Balanced Target Condition (BTC)	$\widehat{P}_j$ (for BTC)	Ψ_j	$P_j = \widehat{P}_j \widetilde{X}_j$ (for BTC)
DB	$q^* W$	$q^* = \frac{\tau(R^*-A)}{\omega-R^*}$	$\tau W \frac{R^*-A}{\omega-R^*}$	$\frac{\omega-R^*}{\omega-R} \frac{R-A}{R^*-A}$	$\tau W \frac{R-A}{\omega-R} \Delta$
AR	$\kappa^*(R-A)W$	$\kappa^* = \frac{\tau}{\omega-R^*}$	$\tau W \frac{R-A}{\omega-R^*}$	$\frac{\omega-R^*}{\omega-R}$	$\tau W \frac{R-A}{\omega-R} \Delta$
NDC	$\tau W \frac{R-A}{\omega-R}$	—	$\tau W \frac{R-A}{\omega-R}$	1	$\tau W \frac{R-A}{\omega-R} \Delta$

Note: The table shows the formula pension $\widehat{P}_j$, the demographic deduction factor Ψ_j and the total pension $P_j = \widehat{P}_j \widetilde{X}_j$ for three variants of PAYG schemes: DB (Defined Benefit), AR (Accrual Rates), NDC (Notional Defined Contribution). The balanced target condition (BTC) has to hold if the system has a balanced budget in the case that all individuals retire at the target retirement age $R = R^*$. The expression in column (3) follows from inserting column (2) into column (1). The values for Ψ_j in column (4) follow from inserting $\widehat{P}_j$ from column (3) into equation (2) and noting that one can write $\widetilde{X}_j = \Psi_j \Delta$ where $\Delta = 1 + \frac{\delta}{2}\left(R - R^*\right)\frac{\omega-A}{R-A}$. Column (5) is the multiple of columns (3), (4) and Δ.

effect of the deductions X_j while for a NDC system the reduction is (primarily) due to the effect of the formula pensions.

The formula pension levels in the defined benefit and the accrual rate system are based on target parameters q^* and κ^*, respectively. It is reasonable to assume that these parameters are fixed in such a fashion that the PAYG system would be balanced in the case when every individual retires at the target retirement age R^* with a target pension P^*.[7] For a constant cohort size N the revenues of the system are in this case given by $I = \tau W(R^* - A)N$ while the expenditures amount to $E = P^*(\omega - R^*)N$. A balanced budget with $E = I$ thus implies $P^* = \tau W \frac{R^*-A}{\omega-R^*}$. For the DB system, this implies a balanced-budget replacement rate of $q^* = \frac{P^*}{W} = \tau \frac{R^*-A}{\omega-R^*}$. Using this relation in the expressions above one can summarize the formula pension level $\widehat{P}_j$ for the three systems as: $\widehat{P}_{\text{DB}}(R, R^*) = \tau W \frac{R^*-A}{\omega-R^*}$, $\widehat{P}_{\text{AR}}(R, R^*) = \tau W \frac{R-A}{\omega-R^*}$ and $\widehat{P}_{\text{NDC}}(R, R^*) = \tau W \frac{R-A}{\omega-R}$. Note that the balanced budget target pension P^* (at $R = R^*$) is the same in all three systems. For a better overview table 1 contains the expressions that have been derived so far in columns (1) to (3).

One can now insert the pension levels for $\widehat{P}_j$ (column (3) of table 1) into equation (2) in order to derive the (approximated) expressions for the budget-neutral deduction factor $\widetilde{X}_j$ for the three systems $j \in \{\text{DB}, \text{AR}, \text{NDC}\}$. It turns out that this approximated deduction factor can be expressed as: $\widetilde{X}_j = \Psi_j \Delta$, where Ψ_j is a "demographic part"

[7]In section 5 I also look at the case where this assumption does not hold true.

that just depends on the demographic and economic variables ω, A, R and R^*, while Δ is a "financing" part that also depends on the discount rate δ.[8] In particular, $\Psi_{\text{DB}} = \frac{\omega-R^*}{\omega-R}\frac{R-A}{R^*-A}$, $\Psi_{\text{AR}} = \frac{\omega-R^*}{\omega-R}$, $\Psi_{\text{NDC}} = 1$ and $\Delta = 1 + \frac{\delta}{2}(R-R^*)\frac{\omega-A}{R-A}$. These results are collected in column (4) of table 1. Column (5) shows that the application of the deduction factor $\widetilde{X}_j$ leads to an identical final pension payment P_j for all three systems.

2.4 Deductions for different discount rates

One can now look at the deductions for various assumptions of the discount rate. At the moment I am not concerned about the budgetary implications of this choice and I am just focusing on the level of deductions that follow from the exact X_j (see equation (1)) or the approximated $\widetilde{X}_j$ (see (2)). In the literature one can find two benchmark assumptions concerning the discount rate which will be discussed below. As a first possibility it is assumed that $\delta = r$, i.e. the discount rate is set equal to the market interest rate. As a second possibility it is argued that the social discount rate should be set to the internal rate of return of a PAYG pension system. In the simple example of this section without economic or population growth the internal rate of return is zero and thus $\delta = 0$. In fact, it is straightforward to show that for a growing economy with $W(t) = W(0)e^{gt}$ and where ongoing pensions are adjusted with the growth rate g equation (2) is unchanged except that now δ has to be substituted by the "net discount rate" $\hat{\delta} \equiv \delta - g$.

The assumption $\delta = 0$ (or $\hat{\delta} = 0$) is a natural starting point which implies that $\Delta = 1$ and also $X_{\text{NDC}} = \widetilde{X}_{\text{NDC}} = 1$. The basic formula of the NDC system $P_{\text{NDC}} = \widehat{P}_{\text{NDC}} = \tau W \frac{R-A}{\omega-R}$ is thus enough to implement the required reduction for early retirement that fullfills the neutrality condition (1). This is different for the two other variants where the pension formula does not suffice to stipulate the necessary reductions even though $\Delta = 1$. In particular, the additional deduction has to be such that the final pension is exactly equal to $P_{\text{NDC}} = \tau W \frac{R-A}{\omega-R}$. For the case of the accrual rate system this means that $X_{\text{AR}} = \frac{\omega-R^*}{\omega-R}$ while for the DB system one gets that $X_{\text{DB}} = \frac{\omega-R^*}{\omega-R}\frac{R-A}{R^*-A}$.

For a positive discount rate $\delta > 0$, however, even a NDC system will not lead to long-run stabilization. It is useful to illustrate the magnitude of these effects for realistic numerical values. In particular, assume that people start to work at the age of $A = 20$, that they die at the age of $\omega = 80$, that the contribution rate is $\tau = 0.25$, the target retirement age $R^* = 65$ and the constant wage $W = 100$. In tables 2 and 3 I show the

[8] The two coefficients are functions of the various variables, i.e. $\Psi_j = \Psi_j(R, R^*, \omega, A)$ and $\Delta = \Delta(R, R^*, \omega, A, \delta)$. For better readability I again leave out the function arguments.

Table 2: Deductions for $R = 64$ and $R^* = 65$

		$\delta = 0$			$\delta = 0.02$			$\delta = 0.05$		
Type j	$\widehat{P}_j$	X_j	x_j(in%)	P_j	X_j	x_j(in%)	P_j	X_j	x_j(in%)	P_j
DB	75.00	0.92	-8.33	68.75	0.90	-9.64	67.77	0.88	-11.81	66.14
AR	73.33	0.94	-6.25	68.75	0.92	-7.59	67.77	0.90	-9.80	66.14
NDC	68.75	1	0	68.75	0.99	-1.43	67.77	0.96	-3.79	66.14

Note: The table shows the actuarial deduction factors X_j, the annual deductions rates x_j (based on the linear relation $x_j = \frac{X_j - 1}{R^* - R}$) and the final pension $P_j(R, R^*) = \widehat{P}_j(R, R^*)X_j$ for three pension schemes and three discount rates. The numerical values are: $A = 20$, $\omega = 80$, $\tau = 0.25$, $W = 100$, $R^* = 65$ and $R = 64$. All cohort members are assumed to reach the maximum age (rectangular survivorship).

Table 3: Deductions for $R = 60$ and $R^* = 65$

		$\delta = 0$			$\delta = 0.02$			$\delta = 0.05$		
Type j	$\widehat{P}_j$	X_j	x_j(in%)	P_j	X_j	x_j(in%)	P_j	X_j	x_j(in%)	P_j
DB	75.00	0.67	-6.67	50.	0.62	-7.70	46.13	0.53	-9.33	40.01
AR	66.67	0.75	-5.00	50.	0.69	-6.16	46.13	0.60	-8.00	40.01
NDC	50.00	1.	0.	50.	0.92	-1.55	46.13	0.80	-4.00	40.01

Note: The table shows the actuarial deduction factors X_j, the annual deductions rates x_j (based on the linear relation $x_j = \frac{X_j - 1}{R^* - R}$) and the final pension $P_j(R, R^*) = \widehat{P}_j(R, R^*)X_j$ for three pension schemes and three discount rates. The numerical values are: $A = 20$, $\omega = 80$, $\tau = 0.25$, $W = 100$, $R^* = 65$ and $R = 60$. All cohort members are assumed to reach the maximum age (rectangular survivorship).

magnitude of the necessary budget-neutral deductions for the case of $R = 64$ ($R = 60$) and three values of the discount rate δ (0%, 2% and 5%).[9] In order to transform the total deduction factor X into an annual (or rather period) deduction rate x there exist two possibilities. As one possibility one can use the continuous-time framework to conclude from $X = e^{x(R^* - R)}$ that $x = \frac{\ln(X)}{R^* - R}$. In existing pension systems, however, the period deductions are typically expressed in a linear way, i.e. $x = \frac{X - 1}{R^* - R}$. In the following tables I show the period deduction rates (in %) based on this linear formula.

All three systems promise a pension of $\widehat{P}(R^*, R^*) = 75$ for a retirement at age $R^* = 65$. For early retirement at $R = 64$ the formula pension is reduced to $\widehat{P}_{\text{NDC}}(R, R^*) = 68.75$

[9] The numbers show X_j, i.e. the exact solutions to equation (1) and not $\widetilde{X}_j$ of the approximated formula (2). The quantitative differences between these two magnitudes are, however, small.

for the NDC system which is the actuarial amount as long as $\delta = 0$. For the accrual rate system, on the other hand, the formula pension is only reduced to $\widehat{P}_{\text{AR}}(R, R^*) = 73.33$ and the system thus needs additional deductions in order to guarantee stability. For the current example the necessary annual deduction rate is 6.25%. For the traditional DB system the annual deduction rate is even larger (8.33%) since there is no adjustment of the pension $\widehat{P}_{\text{DB}}(R, R^*)$. For discount rates above 0 also the NDC needs extra deductions. For $\delta = 0.02$, e.g., the annual deductions are 1.43% and for $\delta = 0.05$ they are 3.79%. For the DB and the AR system the annual deductions also increase by an amount that is somewhat smaller than the extent of δ. If one looks at the even earlier retirement at age $R = 60$ (see table 3) then the results are qualitatively similar. Now the NDC pension is only 50 instead of 75 (for $\delta = 0$) and for the other two systems the annual deductions are somewhat smaller than before.[10]

Summing up, one can conclude that the levels of actuarial deductions depend both on the exact pension formula and on the choice of the social discount rate. For $\delta = 0$ the basic formula of the NDC system is sufficient and no additional deductions are necessary. For the DB and AR systems, however, even for $\delta = 0$ one needs deductions that depend on the demographic structure and on the rules of the pension system. These "demographic deduction factors" are sizable (for our numerical examples between 5% and 8%) and typically larger than the additional deductions that are necessary if one chooses a positive discount rate. In appendix C I show that these conclusions remain valid in a more general framework.

3 Budget-neutral deductions

In the previous sections I have discussed the rates of deduction for different values of the discount rate without looking at the budgetary implications of the various choices. In this section I focus on the appropriate choice to implement a PAYG system that runs a balanced budget. "Budgetary neutrality" requires that retirement before and after the target retirement age does not have an effect on the budget of the pension system in the long run. This requirement implies that one has to consider the *collective* retirement behavior in order to be able to evaluate the budgetary consequences. The look

[10]This is due to the fact that the deductions now have more time to take force and therefore the annual deductions can be smaller. For the same reason it also holds that supplements for late retirement (e.g. $R = 66$, results not shown) are larger than the corresponding deductions for early retirement (e.g. $R = 64$). There is less time to reap the benefits of later retirement and therefore the period supplements have to be higher.

at individual retirement decisions is not sufficient since early retirement of one group might be accompanied by late retirement of another group such that the *average* retirement age stays unchanged. Furthermore, even if the average retirement age in a certain period is below the target age this might still be counterbalanced by higher average retirement ages in later periods. The assessment of budgetary neutrality is thus impossible without the consideration of the intratemporal and intertemporal distribution of retirement ages. Deductions (over and above the demographic part) are only needed insofar as the system has to take out loans in order to finance additional expenditures. If the system can use the intra- and intertemporal variations to provide the necessary funds then these additional financing needs can be reduced or completely avoided.

In the following I discuss this issue for a number of interesting cases. In the first case the distribution of retirement ages is assumed to be stationary over time. For this case it can be shown that a NDC system is always balanced. Since there are no extra financing needs this finding corresponds to an implicit discount rate of $\delta = 0$. In the second case it is assumed that everybody retires at the target age and only one individual of one cohort at a lower age. This one-time-shock scenario represents the simplest example of a non-stationary retirement distribution and is the benchmark case of the related literature. I derive that in this case the correct choice of the discount rate is given by $\delta = r$. In a third section I look at various other non-stationary distributions and calculate the appropriate actuarial deductions for these situations. All examples have in common that they start from a specific distribution and return to the same distribution after an intermezzo of non-stationary periods. It will appear that for many of these situations the actuarial deduction rates are considerably lower than the values of the one-time-shock scenario and often close to zero.[11]

3.1 Set-up and budget

In order to calculate the level of budget-neutral deductions the natural first step is to the define the budget of the pension system. I stick to the simplified model of the previous section, i.e. to a model in continuous time with the assumption of rectangular survivorship where all members of a cohort reach the maximum age ω. The wage is fixed at W and the contribution rate at τ. In appendix C it is shown that the main results also hold in a model with a growing wage level and with an explicit mortality structure. In every

[11] A separate issue is the case where the distribution of retirement ages (and in particular the average retirement age) shifts over time. The discussion of this case is postponed to section 6.2.

instant of time a cohort of equal size N is born. The length of the working life (and thus the number of contribution periods) depends on the starting age and the retirement age. For sake of simplicity I assume that all individuals start to work at age A and are continuously employed up to their individual retirement age R. For the latter I assume that the age-specific probability to retire for generation t is given by $f(a,t)$ for $a \in [A, \omega]$. The cumulative function $F(a,t)$ indicates the fraction of cohort t that is already retired at age a. It holds that $F(A,t) = 0$ and $F(\omega, t) = 1$. In the simple model of this section retirement fluctuations are the only possible source of non-stationarity.

The total (adult) population $Q(t)$ is constant and given by:

$$Q(t) = N(\omega - A). \tag{3}$$

The size of the retired population $M(t)$ can be derived from the following considerations. For a given retirement age R there are individuals of ages $a \in [R, \omega]$ that are in retirement. Their relative frequencies are given by $f(R, t-a)$.[12] Integrating over all possible retirement ages $R \in [A, \omega]$ leads to:

$$M(t) = N \int_A^\omega \left(\int_R^\omega f(R, t-a)\,\mathrm{d}a \right) \mathrm{d}R. \tag{4}$$

The total size of the active population $L(t)$ can be calculated as:

$$L(t) = Q(t) - M(t) = N \left[(\omega - A) - \int_A^\omega \left(\int_R^\omega f(R, t-a)\,\mathrm{d}a \right) \mathrm{d}R \right]. \tag{5}$$

Turing to the budget of the system, total revenues $I(t)$ are given by:

$$I(t) = \tau W(t) L(t). \tag{6}$$

Total expenditures $E(t)$, on the other hand, can be written as:

$$E(t) = N \int_A^\omega \left(\int_R^\omega P(R, a, t-a) f(R, t-a)\,\mathrm{d}a \right) \mathrm{d}R, \tag{7}$$

where $P(R, a, t-a)$ stands for the pension payment of a member of cohort $t-a$. The size of the pension can depend on the payment period t, on the individual's age a and

[12] Note that $f(R, s)$ denotes the retirement density of the cohort born in period s. In period t the mass of individuals who retired at age R and are now a years old is therefore given by $f(R, t-a)$.

also on the time of his or her retirement $R \leq a$. As in the previous section the pension P is the product of the formula pension $\widehat{P}$ and the deduction factor χ. In particular, one can write $P_j(R, a, t-a) = \widehat{P}_j(R, a, t-a)\chi_j(R, a, t-a)$ for $j \in \{\text{AR}, \text{DB}, \text{NDC}\}$. Both the formula pension and the deduction factor might depend on t, a and R. In the following I will concentrate on the NDC system since the other two systems can be transformed into the NDC scheme easily by the use of the demographic adjustment factors Ψ_{AR} and Ψ_{DB} as shown in section 2. For the sake of readability I leave out the subscript "NDC" in the following and thus write $\widehat{P}(R, a, t-a) = \widehat{P}(R) = \tau W \frac{R-A}{\omega - R}$. In the following the deduction factor thus refers to the NDC system. In principle, this deduction factor $\chi(R, a, t-a)$ might differ with respect to time, age and the retirement age. One could, e.g., have a deduction factor that changes from period to period in reaction to the general budgetary outlook. This would be similar to the ABM (Automatic Balance Mechanism) in the Swedish system. Alternatively one might have a situation where the deduction factor is not calculated once and for all at the moment of retirement but changes during the retirement years. This possible age- and time-dependence will of course have implications for the intra- and intergenerational distribution. I abstract from these issues here, however, and focus on deduction factors that are independent of time and age and only differ with respect to the retirement age, i.e. $\chi(R, a, t-a) = X(R)$. For the NDC system (where the demographic part of the deduction factor is given by $\Psi = 1$) one can write $X(R) = (1 + x(R - R^*))$ where x is the time-invariant deduction rate. This is the specification that is employed in the related literature and that also corresponds to the design of real-world deduction rates. One should nevertheless bear in mind that one could also use different and more elaborate specifications for $\chi(R, a, t-a)$.

The deficit in period t is defined as:

$$D(t) = E(t) - I(t) \tag{8}$$

and the deficit ratio as:

$$d(t) = \frac{D(t)}{I(t)} = \frac{E(t)}{I(t)} - 1. \tag{9}$$

A balanced budget in period t thus requires $D(t) = 0$ or $d(t) = 0$. The intertemporal balanced budget constraint between some periods t_0 and t_T, on the other hand, reads as:

$$\int_{t_0}^{t_T} D(t) e^{-r(t-t_0)} \mathrm{d}t = 0, \tag{10}$$

where r is the capital market interest rate that has to be used to finance possible budgetary shortfalls (or at which possible surpluses can be invested). Budget-neutral deductions can then be defined as the value of x such that equation (10) is fulfilled.[13]

In the following I will discuss a number of cases and investigate how the budget-neutral deduction rate x depends on the assumption concerning the collective retirement behavior as captured by the density functions $f(R,t)$.

3.2 Case 1: A stationary distribution of retirement ages

I start with the natural benchmark case of a stationary retirement distribution, i.e. $f(R,t) = f(R)$ and $F(R,t) = F(R)$. The main result is summarized in the following proposition.

Proposition 1

Assume a situation with a constant wage rate W, a constant cohort size N, a constant entry age A, a constant longevity ω, rectangular mortality and a retirement age that is distributed according to the density function $f(R)$ for $R \in [A,\omega]$. In this case a NDC system will be in continuous balance ($D(t) = 0, \forall t$) without the need of additional deductions ($X(R) = 1$ or $x = 0$).

Proof. In order to see this I first assume that the proposition is correct (i.e. $x = 0$) and then show that this in fact leads to a balanced budget with $D(t) = 0$. To do so one can insert the NDC pension $P(R) = \tau W \frac{R-A}{\omega-R}$ (assuming $x = 0$) into (7) which leads to

$$\begin{aligned} E(t) &= N \int_A^\omega \left(\int_R^\omega \tau W \frac{R-A}{\omega - R} f(R)\,\mathrm{d}a \right) \mathrm{d}R \\ &= \tau W N \int_A^\omega (R-A) f(R)\,\mathrm{d}R = \tau W N(\overline{R} - A), \end{aligned}$$

where $\overline{R} \equiv \int_A^\omega R f(R)\,\mathrm{d}R$ stands for the average retirement age. This is the same as total revenues since

$$\begin{aligned} I(t) &= \tau W L(t) = \tau W N \left[(\omega - A) - \int_A^\omega \left(\int_R^\omega f(R)\,\mathrm{d}a \right) \mathrm{d}R \right] \\ &= \tau W N \left[(\omega - A) - (\omega - \overline{R}) \right] = \tau W N(\overline{R} - A) = E(t). \blacksquare \end{aligned}$$

[13] If the period-by-period balancing condition $D(t) = 0, \forall t$ were used instead of the intertemporal balanced budget condition (10) one would need a time-varying deduction factor $\chi(R, t-a)$ (or possibly $\chi(R, a, t-a)$) in order to guarantee the balanced budget.

For a stationary distribution of retirement ages $f(R)$ a pure NDC system is thus balanced in every period ($D(t) = 0, \forall t$). There is no need for loans to finance the early retirement of some individuals, the capital market interest rate is irrelevant and extra deductions are unnecessary ($x = 0$). Using the results of section 2 (see table 1) this also implies that the appropriate discount rate for the standard deduction equation (1) is $\delta = 0$.

The intuition behind this result is straightforward. The system needs money to finance the pension of the early retirees with a $R^L < \overline{R}$. This is available, however, since in the previous periods the early retirees did not get the full pension that would be paid for retirement at the average age $\overline{R}$ (i.e. $P = \tau W \frac{\overline{R}-A}{\omega-\overline{R}}$) but rather the smaller pension $P = \tau W \frac{R^L-A}{\omega-R^L}$. A similar argument holds for the late retirees where their higher pension can be financed by the extra contributions of the late retirees of future generations.

3.3 Case 2: A one-time shock in retirement ages

This case is dominant in the related literature on actuarial deductions (Börsch-Supan 2004, Werding 2007, Gasche 2012). In particular, the situation is based on the thought experiment that everybody retires at the target retirement age R^* except one individual who chooses a lower retirement age.[14] To be more precise, I assume that there is a small mass θ of members of cohort $\hat{t}$ who retire at $R^L < R^*$. All other individuals retire at the target age. The question is how to choose the deduction factor $X(R^L)$ (or the deduction rate x) such that the intertemporal budget condition (10) ($\int_{t_0}^{t_T} D(t)e^{-r(t-t_0)}\mathrm{d}t = 0$) is fulfilled (for $t_0 < \hat{t} < t_T - \omega$).

The first thing to note is that in all periods before $\hat{t} + R^L$ the deficit is balanced. From periods $(\hat{t} + R^L)$ to $(\hat{t} + R^*)$ the revenues of the system are lower than normal due to the early retirement of the deviating group of mass θ. For these periods the deficit $D(t)$ is further increased due to the fact that the early retirees already receive a pension payment $P^L = \widehat{P}^L X(R^L) = \tau W \frac{R^L-A}{\omega-R^L} X(R^L)$ which would not be the case had they stayed employed until the target retirement age R^*. On the other hand, the expenditures of the system are lower than normal for the time periods between $(\hat{t} + R^*)$ and $(\hat{t} + \omega)$ due to the fact that the pension of the early retirees is lower than the target pension. After the early retirees have died in period $(\hat{t} + \omega)$ the pension system is back to the normal mode with a continuous balance of $D(t) = 0$. The intertemporal budget condition (10) only

[14] In fact, it is not necessary that everybody retires at the target age but only that everybody retires at the *same* age.

involves non-zero values for these exceptional periods and can thus be written as:

$$\theta \int_{\hat{t}+R^L}^{\hat{t}+R^*} \tau W e^{-r(t-(\hat{t}+R^L))}\,\mathrm{d}t + \theta \int_{\hat{t}+R^L}^{\hat{t}+R^*} \widehat{P}^L X(R^L) e^{-r(t-(\hat{t}+R^L))}\,\mathrm{d}t$$
$$-\theta \int_{\hat{t}+R^*}^{\hat{t}+\omega} \left(P^* - \widehat{P}^L X(R^L)\right) e^{-r(t-(\hat{t}+R^L))}\,\mathrm{d}t = 0.$$

Choosing $\hat{t} = 0$ and canceling θ one can observe that this is exactly the same expression as the standard deduction equation (1) with the choice of a discount rate $\delta = r$. The necessary deduction factor is then given by the formula in (2), i.e. by $X(R^L) = \Delta = 1 + \frac{r}{2}\left(R^L - R^*\right)\frac{\omega - A}{R^L - A}$ (see table 1).

In this one-time-shock scenario the size of the interest rate has an impact on the budget-neutral deduction rate since the pension system has to take out a loan at the interest rate $r > 0$ in order to deal with the financial consequences of the early retirement decisions.[15]

3.4 Case 3: Further non-stationary distributions of retirement ages

Case 2 is the simplest case of a non-stationary distribution. It is based, however, on a highly stylized scenario and it would be misleading to neglect other, arguably more plausible scenarios. In the following I discuss two examples.

Two-point distribution For non-stationary patterns of retirement it is typically not possible to derive analytical solutions and one has to revert to numerical simulations. There exists, however, one particularly simple distribution that can be solved analytically and is thus useful to fix ideas. In particular, assume that up to cohort $\hat{t}$ each individual either chooses a low retirement age R_1^L or a high age R_1^H, with relative frequencies p_1 and $(1-p_1)$, respectively. The average retirement age per cohort is thus given by $\overline{R}_1 = p_1 R_1^L + (1-p_1)R_1^H$. From cohort $\hat{t}$ on there is a shift in retirement behavior and now a fraction p_2 retires at age $R_2^L > R_1^L$ and a fraction $(1-p_2)$ at age $R_2^H < R_1^H$. The aggregate retirement age per cohort, however, is assumed to stay the same, thus $\overline{R}_2 = p_2 R_2^L + (1-p_2)R_2^H = \overline{R}_1$. The system is balanced before period $\hat{t} + R_1^L$ and after period $\hat{t} + \omega$ but in-between there will be a number of periods with budget surpluses and deficits.

[15] This is, e.g., the argument used by Werding (2007) to justify the use of market interest rates to calculate budget-neutral deductions.

In particular, the early retirees of cohort $\hat{t}$ retire later than the early retirees of previous cohorts ($R_2^L > R_1^L$). This means that the system has a number of periods with higher revenues and lower expenditures in which it runs a surplus. On the other hand, later on there will be periods with a larger number of retirees than before due to the fact that the late retirees leave the labor market sooner ($R_2^H < R_1^H$). In these periods the system will face more expenditures, less revenues and will thus run a deficit. The challenge is to choose a deduction rate x such that the present value of the sum of these surpluses and deficits is zero, i.e. such that the intertemporal balanced budget condition (10) is fulfilled. This budget-neutral deduction rate will clearly depend on the size of the interest rate r. In appendix A I show that in the case with $r = 0$ and $x = 0$ the present value $\overline{D} = \int_{\hat{t}+R_1^L}^{\hat{t}+\omega} D(t)e^{-r(t-(\hat{t}+R_1^L))}\mathrm{d}t$ comes out as $\overline{D} = \frac{1}{2}\tau W N(\overline{R}_1 - \overline{R}_2)(\omega - A)$. Since in the example I have assumed that $\overline{R}_1 = \overline{R}_2$ it follows that $\overline{D} = 0$ and thus with $r = 0$ and $x = 0$ the surpluses and deficits just offset each other. For positive values of r this is, however, no longer true and one needs a non-zero deduction rate x. For example, with $\tau = 0.25$, $W = 100$ and $R_1^L = 60$, $R_1^H = 70$, $R_2^L = 65$, $R_2^H = 65$, $p_1 = p_2 = \frac{1}{2}$ and thus $\overline{R}_1 = \overline{R}_2 = 65$ one can calculate that $x = -0.0057$ (for $r = 0.02$) and $x = -0.014$ (for $r = 0.05$). These deduction rates are thus larger than in the benchmark NDC case (where $x = 0$) but also considerably smaller than the standard deductions based on equation (1) where they come out as $x = -0.0133$ (for $r = 0.02$) and $x = -0.0333$ (for $r = 0.05$). The use of these larger deduction rates would lead to a permanent surplus of the system

The one-time shift in the two-point distribution is, however, again a rather special case of a non-stationary development. There exist at least two reasons why it is not a particularly useful benchmark for realistic scenarios. First, the retirement ages follow a deterministic process and second the average retirement age of all retirees does not stay constant (for the example above it increases from 63.3 to 65). The following stochastic example does not suffer from either of these problems.

Random fluctuations: In order to study more complicated non-stationary distributions one has to revert to numerical simulations in a discrete-time framework, since analytical solutions are no longer possible. Appendix B contains a detailed description of the discrete-time model. As a benchmark scenario I look at a situation where the cohort-specific retirement densities $f(R,t)$ are random draws from a stable distribution $f^*(R)$. In the appendix I discuss the results of an example where this stable retirement distribution is triangular between the ages 60 and 70 with a mean at $\overline{R} = 65$ that is also the target age R^*. The parameters of the simulation are chosen in such a way that fluctuations in the retirement age roughly correspond to real-world data. For each simulation I calculate the deduction rate x that solves the discrete-time equivalent of equation (10) and I verify that this value manages to keep the budget in balance. Figure A.2 in the appendix illustrates this for one specific simulation. Over one-hundred simulation runs the average of these budget-neutral deduction rates x is close to zero. In particular, it comes out as $\overline{x} = 0.0002$ with a standard deviation of 0.003.

This shows that the result of proposition 1 also holds approximately true for time-variant retirement distributions. A pure NDC system with only minimal additional deductions will be compatible with a stable long-run budget as long as the retirement ages fluctuate around a stationary target distribution.[16]

3.5 Summary

In this section I have used various examples to emphasize a crucial point: the level of budget-neutral deductions depends on the assumptions concerning the *collective* retirement behavior. For a stationary retirement distribution the formula pension of a standard NDC system is sufficient to guarantee a balanced budget and there is no need for additional deduction. This corresponds to the choice of a discount rate $\delta = 0$ in the commonly used equation (1). For the often used thought experiment of a one-time-shock there are additional financing needs and the formula pension has to be amended by a deduction rate that follows from equation (1) by setting $\delta = r$. Both of these benchmark scenarios are arguably rather stylized. Neither do policymakers face a population with a completely stationary retirement behavior nor do they observe only a few individuals that deviate from the statutory retirement age. In fact, the second scenario is not an ultimately convincing benchmark case since it cannot be extended over time. In the year after the single early retiree left the labor market there will no longer be a situation where all individuals have an identical retirement age (which has been the initial situation of the thought experiment). One could assume that a constant fraction of each cohort chooses the early retirement age but this would then correspond to the alternative situation of a retirement distribution that is stationary over time. If one looks at empirical data (see appendix B) then it seems to be the case that—absent radical policy reforms—the retirement distribution is rather constant and changes only slowly over time. The scenario of a stable target distribution with random fluctuations around this distribution thus seems to be a better

[16] In fact, setting $x = 0$ from the beginning and calculating the revenues and expenditures for the 100 simulation runs also leads to a budget that is (almost) balanced on average, however with a considerably larger standard deviation of budgetary outcomes across the different simulations.

approximation of real-world behavior. The results of section 3.4 then suggest that the use of a NDC system without additional deductions ($x = 0$) will guarantee an approximately balanced budget in this situation.

The entire discussion so far has, however, been built on the assumption that there are no long-run shifts in retirement behavior like an increase in average retirement age. In section 6 I will come back to this issue. I will argue that cases with long-run shifts are certainly not unrealistic but that the deduction rates are the wrong parameter to deal with this phenomenon.

4 Excessive deductions

Even though a deduction rate of $x = 0$ (with a corresponding discount rate $\delta = 0$) is sufficient for a stable budget in the case of stationarity (see proposition 1) it is nevertheless interesting to look at a situation where despite this fact the system chooses an additional deduction rate $x < 0$ (or a positive discount rate $\delta > 0$). In order to study this in more detail one can use the expression from section 2 for the NDC-pension given in column (5) of table 1:

$$P = \widehat{P}\widetilde{X} = \tau W \frac{R-A}{\omega - R}\left(1 + \frac{\delta}{2}\left(R - R^*\right)\frac{\omega - A}{R - A}\right). \tag{11}$$

Inserting this expression together with the assumed stationary retirement density $f(R)$ into equation (7) leads to total expenditures $E(t) = \tau W N\left(\overline{R} - A + \frac{\delta}{2}(\omega - A)(\overline{R} - R^*)\right)$. Total revenues (6), on the other hand, are given by $I(t) = \tau W N(\overline{R} - A)$. From this one can calculate the deficit as

$$D(t) = \tau W N \frac{\delta}{2}(\omega - A)\left(\overline{R} - R^*\right). \tag{12}$$

Equation (12) confirms the results from above that the budget of the system with a stationary retirement distribution is in permanent balance if one chooses a discount rate of $\delta = 0$. For the case of a positive discount rate $\delta > 0$ one has to distinguish between two cases. If the target retirement age R^* is set equal to the average actual retirement age $\overline{R}$ then the budget of the system is still in balance, while for $R^* \neq \overline{R}$ this is no longer true. The implications of these two cases are studied in the following.

Balanced budget with higher deductions Equation (12) implies that a positive discount rate is still compatible with a permanently balanced budget as long as $R^* = \overline{R}$.

The positive discount rate is, however, not necessary for budgetary reasons since the balance would materialize for *any* value of δ. This means that the higher pensions for the later retirees are paid by the deduction of the early retirees. Whether this is a reasonable and fair property depends on the preferences and the objectives of the system. If, e.g., the main reason for early retirement is seen as a preference for leisure and if (for whatever reason) the policy-makers strive for a higher average retirement age it looks defendable to implement such an unnecessary scheme of intragenerational redistribution. If, on the other hand, early retirement is due to bad health or harsh working conditions it might be deemed unfair that those individuals get an additional punishment for their early retirement by paying for the augmented pensions of their more fortunate peers.

Permanent deficits or surpluses with higher deductions An even more severe deviation from the benchmark case with $\delta = 0$ occurs if the discount rate is positive *and* the target retirement age is set above the average retirement age ($R^* > \overline{R}$). In this case equation (12) indicates that the budget will show a permanent surplus ($D(t) < 0$) while there will be a permanent deficit ($D(t) > 0$) in the reverse case ($R^* < \overline{R}$). To give a numerical example set $A = 20$, $\omega = 80$, $R^* = 65$ and $\overline{R} = 63$. If the social planner chooses a discount rate of $\delta = 0.05$ then the deficit ratio $d(t) \equiv D(t)/I(t)$ comes out as $d(t) = -0.07$. This means that every period the expenditures are 7% lower than the revenues and the system is permanently in surplus.

It is not straightforward to decide whether a situation with a permanent surplus or a permanent deficit is reasonable. In order to do so one would need to specify the optimal size of the PAYG pension system which depends—inter alia—on individual and social preferences, on the economic environment and also on the history of the system. Such an analysis is beyond the scope of the present paper. In fact, in the framework of the simple model presented so far there exists no prima facie reason for a PAYG system since a funded pension system with $r > g$ (where in the simple model $g = 0$) will provide a higher rate of return that does not have to be weighted against potentially higher risk (due to the assumption of complete certainty). In reality, however, the rationale behind the introduction of a PAYG system is typically based on concerns about myopic behavior, intergenerational smoothing, poverty prevention and risk-sharing. As a short-cut one could assume that these concerns have been taken into account in the original design of the PAYG pillar and that the optimal (i.e. socially preferred) size of the system corresponds to the actual revenues at the time of its introduction, i.e. to $I(0) = \tau W N(\overline{R} - A)$. In this case, however, it would not make sense to choose a target level $R^* > \overline{R}$ and a deduction

$\delta > 0$ since this would lead to a permanent surplus and thus to a shrinkage of the PAYG system as compared to its optimal size. In particular, the pension for $R^* > \overline{R}$ and $\delta > 0$ would be lower for individuals with $R < R^*$ than for the benchmark case with $\delta = 0$. The early retirees would thus pay for the partial dismantling of the PAYG system. They are treated like the transition generations after the abolishment of a PAYG system who also have to contribute twice (once for the public pension system and once for their private savings).

These are not academic reflections in the framework of a stylized model but they form the background of regular and sometimes rather heated debates on the appropriate level of deductions in many countries. In Germany, some authors have argued for considerably higher deduction rates than the current value of 3.6% in order to implement an incentive-compatible scheme (Börsch-Supan & Schnabel 1998, Börsch-Supan 2004, Werding 2007). Others have countered that it is problematic on normative grounds to choose deductions that are higher than required for budget neutrality just in order to achieve general political goals (Ohsmann et al. 2003).

5 Extensions without long-run changes

So far I have focused on a simple economic and demographic set-up in order to derive the main results in an intuitive and often analytical manner. In order to achieve this I had to abstract from many interesting aspects that are important for real-world systems. In this section I offer a number of extensions of the basic set-up. In particular, I discuss how the main results are affected in a set-up with growing wages and mortality (section 5.1), with heterogeneity of entry ages, employment histories and wages (section 5.2) and with life expectancies that are correlated with individual incomes (section 5.4). In addition I also look at the case where a pension system is unbalanced by design. (section 5.3). The main conclusion from these extensions is that a stationary distribution of retirement ages is associated with a situation where the level of actuarial deductions is independent of the market interest rate. The existence of heterogeneity might, however, lead to problematic distributional outcomes. In particular, if the system is designed in a way that for some (or for all) individuals the sum of pension payments exceeds the sum of contributions if they retire at the target age then it is not clear whether or to which extent the actuarial deductions should preserve this "bonus".

All of these results are, however, derived in an environment where the crucial demographic variables are assumed to be stationary. In section 6 I focus on scenarios that also involve time-variability (e.g. increasing mean retirement age or increasing life expectancy). For these constellations it is not straightforward to calculate actuarial deductions since the system might become unbalanced even if all people retire at a specific target age. In these situations one has to adjust the basic formulas of the pension system in order to keep it balanced. This involves difficult issues that go beyond the determination of deductions.

5.1 Mortality and growing economy

In sections 2 and 3 I have used a model with rectangular mortality and with constant wages. In appendix C I look at a set-up with non-rectangular survivorship $S(a)$ (where $S(0) = 1$ and $S(\omega) = 0$) and a growing wage $W(t)$. I show how the main formulas have to be adapted. Furthermore, I demonstrate that all main findings continue to hold in this more general framework. In particular I show that for a stationary retirement distribution (i) a NDC system is stable without the use of additional deductions or supplements; (ii) the DB and AR systems are also compatible with balanced budgets if they are augmented by demographic deduction factors that are independent of the market interest rate; (iii) the discount rate that corresponds to these budget-neutral deduction factors is given by the internal rate of return (i.e. now the growth rate of wages); (iv) choosing a higher discount rate might still be associated with a balanced budget if the target retirement age is equal to the average actual retirement age.

Quantitatively, I show that the actuarial deductions are lower for non-rectangular survivorship than for the rectangular case (as reported in tables 2 and 3). In particular, I calculate deduction rates for the assumption of a Gompertz mortality model with plausible parameter values. In the case where the discount rate equals the internal rate of return of the PAYG system (i.e. $\delta = g$) the actuarial deductions for the DB system for a retirement at the age of 64 is 7% while it is 4.9% for the AR system (see tables A.2 and A.3 of appendix C). This is smaller than the corresponding rates for rectangular survival where they have been calculated as 8.33% and 6.25%, respectively. For larger discount rates these difference shrink but are still present as documented in the appendix.

5.2 Additional heterogeneity

The real-world is more complex than reflected in the frameworks used so far. Individuals differ along many more dimensions including fertility, mortality, work history and wages.

It can be shown that the main results of sections 2 and 3 will continue to hold in a set-up that allows for heterogeneity in labor market entry age and in the average lifetime wage. If these variables follow a stationary distribution then one can use the same arguments as above to conclude that a pure NDC system without additional deductions is compatible with a stable budget. In fact, one can regard the formulation with fixed A and W as referring to one specific constellation. Since the pure NDC system leads to a balanced budget for this (as for any other) specific subgroup one can conclude that also the aggregate budget will be in balance. Appendix D shows this in more detail. Furthermore, following the arguments of section 3.4 one would guess that fluctuations around this stable joint distribution should also be compatible with an approximately balanced budget.

5.3 Budget deficits in the steady state

So far I have looked at situations in which the pension system had a balanced budget in the envisioned benchmark situation where everybody retires at the target age R^*. I have invoked this "balanced target condition" (BTC), e.g., in table 1. Real-world systems, however, are often based on unbalanced parameter constellations. It turns out that in this case the appropriate deductions are still independent of the market interest rate (for stationary retirement distributions) but that their exact determination might have an effect on the interpersonal distribution. In order to discuss this issue I now assume that there exists an "unbalanced NDC system" that promises a pension $P = \tau W \overline{\eta} \frac{R-A}{\omega - R}$. For $\overline{\eta} > 1$ the system is, e.g., more generous than the benchmark NDC system. It can be shown that in this case the deficit ratio for the benchmark with $R = R^*$ is given by $d = \overline{\eta} - 1$. Each individual with $R = R^*$ receives a larger sum of pension benefits than the total contributions paid into the system. The absolute level of this "bonus" is given by $B(R^*) = \tau W(R^* - A)(\overline{\eta} - 1)$. The deficit of the system has to be covered from the general budget since otherwise the debt level would explode (for $\overline{\eta} > 1$). It is thus reasonable to assume that this benchmark deficit is accepted by the policy-maker. The deductions and supplements for early and late retirement should be determined in a manner such that they lead to the same deficit. It turns out that there are at least two possibilities to achieve this goals. These two variants have, however, different distributional implications.

In appendix D.1 I show that following the same steps as in section 2.1 (and setting $\delta = 0$) leads to the deduction factor $X = 1 + \frac{R^* - R}{R - A} \frac{\overline{\eta} - 1}{\overline{\eta}}$. For a stationary distribution of retirement ages the use of this deduction factor implies a deficit ratio of $d = (\overline{\eta} - 1) \frac{R^* - A}{\overline{R} - A}$. As long as $R^* = \overline{R}$ the use of X thus leads to a deficit ratio that is equal to the benchmark of $d = \overline{\eta} - 1$. The same deficit ratio can, however, also be achieved with the use of $X = 1$. The difference between these two policies is that in the first case the "bonus" to each individual is identical and given by $B^* = B(R^*) = \tau W(R^* - A)(\overline{\eta} - 1)$ while in the second case it comes out as $B(R) = \tau W(R - A)(\overline{\eta} - 1)$.[17] The choice of $X = 1$ thus leads to a situation in which the bonus increases in R, i.e. late retirees get a higher addition to the standard NDC pension than early retirees. On the other hand, in this case the bonus *relative* to total contributions is the same for everybody and given by $\frac{B(R)}{\tau W(R-A)} = \overline{\eta} - 1$. It is not obvious which of the two variants is prima facie more reasonable or more equitable. It seems, however, to be more difficult to find justifications for the second variant that provides a larger bonus in absolute terms for late retirees.

An interesting example for a situation with $\overline{\eta} > 1$ is the case where ongoing pensions are not adjusted with the real growth rate $g(t)$ (as in the general model in appendix C) but only with the rate of inflation. This has two implications. First, the NDC system now is no longer balanced in the benchmark case with $R = R^*$. Second, the bugdet-neutral deduction factors also change similar to the example discussed above. In appendix D.3 I discuss this further and provide numerical examples.

5.4 Socio-economic differences in life expectancy

An unbalanced budget might also arise in a situation where life expectancy and income are correlated. There exists considerable evidence on the correlation between mortality and various socio-economic indicators like education, income or wealth (Chetty et al. 2016). This raises difficult problems for the design of sustainable and fair pension systems which are beyond the scope of this paper (cf. Breyer & Hupfeld 2009). Here I only want to briefly sketch the implications for the derivation of budget-neutral deductions. The main issue is that the standard NDC system uses *average* life expectancy to calculate pension benefits while total benefits depend on *individual* longevity which is unknown at the moment of retirement. This distinguishes longevity from the other variables that enter the pension formula (like contribution years, average life-time wages etc.) that are all known when calculating the first pension. If life expectancy were uncorrelated with all these other pension-relevant variables then differential mortality would not cause budgetary problems and it would not necessarily be problematic from the viewpoint of (ex-ante) fairness. In

[17] It should be noted that for $\overline{\eta} > 1$ early retirement leads to a *supplement* (i.e. $X > 1$ for $R < R^*$). The reason for this is that the standard NDC pension is lower for early retirement and thus the multiplicative factor $\overline{\eta}$ is applied to a smaller "base" which would lower the total "bonus" to the early retiree. The deduction factor compensates for this reduction.

reality, however, life expectancy is correlated with many variables (in particular with income) which has severe consequences for the budget and for fairness considerations as is elaborated in appendix D.2. In particular, the correlation between life expectancy and income leads to a situation where the NDC system is *not* balanced (even not if everybody chooses $R = R^*$). In other words, it can be described as a situation with $\overline{\eta} > 1$. On the other hand, however, the "bonus" is different for different individuals and groups with higher life expectancy are favored by the pension system. It is not clear how the appropriate deductions should be determined in this case. If the initial situation favoring the long-lived is regarded as unfair then the formula pension should be adapted in order to account for this deficiency. If the allocation of the initial situation is accepted then the appropriate deductions should implement a system where early retirement leaves the budget of the system and the distributional allocation unchanged when compared to the benchmark situation. This implies that the deductions take account of the fact that early retirees among the group of long-lived would receive a smaller "bonus" from the system. The use of these deduction factors will then not provide a balanced budget (since this depends on the correlation between life expectancy and other variables) but it will neutralize the impact of early or late retirement on the budget. As shown in appendix D.2 the neglect of these corrected deductions might not only have distributional consequences but also lead to additional budgetary problems. Pushing these difficult issues aside it is, however, still the case that the deduction rates are independent of the market interest rate for a stationary distribution of retirement ages.

6 Extensions with long-run changes

So far I have concentrated on a stable demographic environment (constant cohort sizes, constant mortality) and on stable retirement patterns where the retirement ages either followed a stationary distribution $f(R)$ (section 3.2) or fluctuated around a stationary distribution (sections 3.3 and 3.4). In fact, the concept and derivation of actuarial deductions implicitly requires such a stable reference point against which budget-neutrality can be measured. If there are shifts in the long-run environment then the entire set-up to derive deductions as presented in equation (1) is indeterminate. It is, e.g., unclear how and whether the target retirement age R^* should be adjusted if the average retirement age shifts over time. In general, the reaction to long-run shifts in the demographic environment is the responsibility of the basic formulas of the pension system and not the task of the deduction rates. The optimal design of these adjustment mechanisms is a difficult issue and the answer depends on the causes of the shift, the degree to which individuals and cohorts can or should be held responsible for the changes and on principles of intra- and intergenerational risk-sharing and fairness. The determination of budget-neutral deductions is a separate (and more modest) task that can only be addressed once the adjustment mechanisms to long-run changes shifts have been specified. This issue is briefly discussed in section 6.1 while in section 6.2 I deal with changes in the average retirement age.

6.1 Long-run demographic shifts

If there are long-run shifts in the demographic environment then actuarial deductions can only be calculated after it has been specified how the benchmark system remains in balance in presence of these shifts. This is most easily done in the case of continuous changes in the demographic structure. If, for example, the cohort size increases in an exponential manner (i.e. $N(t) = Ne^{nt}$) then the internal rate of return of the PAYG system is given by $g+n$. It can be shown that if this rate of return is used as the notional interest rate to calculate the NDC pension and as the discount rate δ then the implied budget-neutral deductions are the same as the ones derived above. In Knell (2017*b*) I discuss the parallel case of a continuously increasing life expectancy and show that it also leads to an increase in the internal rate of return that can be used to implement a stable NDC system.

For non-continuous demographic (and economic) changes the situation is more difficult. As has been shown by Valdés-Prieto (2000), the basic NDC formulas are unable to provide automatic financial equilibrium in this situation. There exist various possibilities how to amend the basic system in order to guarantee financial stability. In Sweden, e.g., the "Automatic Balance Mechanism" (ABM) stipulates changes in the notional interest rate and the adjustment rate as a reaction to budgetary imbalances and thus involves both active workers and retirees (Settergren 2001). In Germany, on the other hand, the "sustainability factor" triggers changes in both the contribution rate and the replacement rate in order to keep the balance of the system in balance. These adjustment mechanisms have different implications for the inter- and intragenerational distribution and for risk-sharing. The choice between different adjustment mechanisms is a complex issue that depends on social preferences and political constraints. It has to be stressed, however, that one should not mix the adjustment to long-run demographic changes with the determination of appropriate budget-neutral deductions for early retirement. While the first point involves difficult questions, the latter is a rather straightforward technical issue.

6.2 Permanent change in the average retirement age

The second challenging long-run shift involves a permanent change in the average retirement age $\overline{R}$ (even if the demographic structure stays constant). Long-run changes in average retirement also lead to budgetary imbalances that cannot be corrected by the use of standard rates of deduction.

This can be seen by looking at a slight variant of the basic one-time shock scenario. While in the basic scenario there are only some members of *one* cohort who choose a different retirement age R', it is now assumed that from time $\hat{t}$ onwards *all* cohorts increase their retirement age from R^* to R'. In appendix A.1 I look at this case in detail. It can be shown that the standard deductions based on equation (1) are not sufficient to guarantee budgetary balance. For example, with $A = 20$, $\omega = 80$ and a shift from $R^* = 65$ to $R' = 70$ this standard deduction rate would be given by 1.2% (for r=0.02). This, however, would lead to a massive surplus of the pension system. The deduction rate (and also supplement rate for late retirement) that balances the pension system at some point in time is in this case given by the much larger value of 9.4%. After the system has achieved this balance one would, however, either reduce the deduction rate again to $x = 0$ or change the target retirement age to the new common value of $R^* = R' = 70$.

The reason why this permanent increase in the average retirement age requires such a huge deduction rate in order to balance the budget is the following. The situation can be viewed as an extension of the PAYG system since in the new stationary situation the total steady state revenues and expenditures have increased. The transition thus can be compared to the introduction of a new PAYG system that is associated with windfall gains that can be distributed among the insured population according to some chosen mechanism. A reduction in the average retirement age, on the other hand, corresponds to a downsizing of the PAYG system and leads to transition costs that have to be borne by some cohorts. In fact, also the standard thought experiment of a one-time shock could be interpreted as a a small reduction of the PAYG system that is, however, only temporary. In this case it seems straightforward to allocate the arising transition costs to the group of early retirees who have deviated from the target retirement age and have caused the budgetary shortfall. It should be noted, however, that this allocation is not the only possibility and already in this case one could also find arguments to justify a sharing of the costs between other members of the retiring or already retired cohorts (e.g. as part of an ex-ante risk-sharing arrangement). For situations with a permanent change in the average retirement age the choice of the adjustment strategy has to be scrutinized even more carefully. As shown above, the standard deduction rates are not sufficient to provide for a balanced budget in the presence of long-run shifts in the retirement age. In general one will need an approach with time-varying adjustment factors (and also time-varying target retirement ages) and the choice of the adjustment mechanism has consequences for the intra- and intergenerational distribution of the windfall gains and losses. It will depend on various considerations, e.g. on an assessment to which degree individuals can be held "responsible" for their early or late retirement. This issue goes beyond the scope of this paper. In general, there does not exists an unequivocally best approach towards this problem and different countries have chosen different strategies as discussed above. The Swedish ABM strives for intertemporal budgetary balance while the German sustainability factor aims for a continuous balance.

7 Conclusions

Annual pensions for early retirement have to be lower than pension payments at the target retirement age. But how much lower in order to keep the budget of a PAYG system in balance? In this paper I have provided an answer to this question that is important for every pension system. I have shown that the answer depends on two crucial issues. First, it depends on the nature of the pension system. The rate of deduction can be lower in systems where the formula pension is already leading to a reduction in pension benefits (like in NDC or AR systems). Second, the level of budget-neutral deductions also depends on the collective retirement behavior over time. In particular, I have shown that a NDC system is stable just by following the pension rule without the need for any further deductions if the retirement distribution is stable or if it fluctuates around a stationary distribution. The assumption of collective retirement behavior has implications for the associated choice of the discount factor. In situations with a stable retirement distribution it can be chosen to be equal to the internal rate of return. This is in contrast to the benchmark scenario in the literature that is based on a one-time shock, i.e. on a constellation where everybody retires at the target age and only one individual at a lower age. In this case it is justified to discount future payment streams with the market interest rate. I have also shown that under certain assumptions (e.g. that the average retirement age is equal to the target age) a higher discount rate than necessary is also compatible with a balanced budget. It is not clear, however, whether the use of deduction rates that are above the requirements for budgetary balance are reasonable. They involve excessive punishments for early retirement and excessive rewards for later retirement with probably unintended and undesired implications for the intragenerational redistribution.

For future research it would be interesting to use numerical methods to calculate the budget-neutral deductions in larger models that include many of the economic, demographic and policy details of real-world systems. Furthermore, the analysis in this paper has been based on the assumption of exogenous retirement behavior. Retirement behavior, however, is of course also influenced by the pension rules including the rates of deduction. Since the budget-neutral deduction rates depend on current and future retirement behavior while the latter will react to the deductions one might expect interesting interactions.

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Appendices

A Distributions with a one-time shift

In this appendix I present two simple examples of a distribution of retirement ages that still allows for closed-form solutions (or at least for numerical solutions in a continuous time framework). The first case is the degenerate case of a distribution where every member of a cohort retires at the same age while the second case corresponds to a two-point distribution.

A.1 One-value distribution (section 6.2)

In the first example I assume that up to cohort $\hat{t}$ all members of the cohort (with total size N) retire at age R_1 while from then on they retire at the later age $R_2 \geq R_1$. Before the occurrence of the shock the NDC pension system is in continuous balance. This changes, however, after the cohort of the shock period enters retirement. This can be discussed in more detail by looking at the size of the retired population $M(t)$ and total expenditures $E(t)$. For the definitions one has to distinguish four intervals. Before time $\hat{t} + R_1$ and after period $\hat{t} + \omega$ one is in a steady state situation.

$$M(t) = \begin{cases} N(\omega - R_1) & \text{for } t \leq \hat{t} + R_1, \\ N(\omega - (t - \hat{t})) & \text{for } \hat{t} + R_1 \leq t \leq \hat{t} + R_2, \\ N(\omega - R_2) & \text{for } \hat{t} + R_2 \leq t \leq \hat{t} + \omega, \\ N(\omega - R_2) & \text{for } t \geq \hat{t} + \omega. \end{cases}$$

$$E(t) = \begin{cases} N(\omega - R_1)P_1 & \text{for } t \leq \hat{t} + R_1, \\ N(\omega - (t - \hat{t}))P_1 & \text{for } \hat{t} + R_1 \leq t \leq \hat{t} + R_2, \\ N((\omega - (t - \hat{t}))P_1 + ((t - \hat{t}) - R_2)P_2) & \text{for } \hat{t} + R_2 \leq t \leq \hat{t} + \omega, \\ N(\omega - R_2)P_2 & \text{for } t \geq \hat{t} + \omega, \end{cases}$$

where the pension levels are given by:

$$P_j = \tau W \frac{R_j - A}{\omega - R_j}(1 + x(R^* - R_j)),$$

with $j \in \{1,2\}$ and x is the deduction rate that is applied between periods $\hat{t} + R_1$ and $\hat{t} + \omega$.

The expressions for the size of the labor force $L(t)$ and the total revenues $I(t)$ are simpler:

$$L(t) = N(\omega - A) - M(t), \tag{13}$$

$$I(t) = \tau W L(t). \tag{14}$$

The deficit ratio is given by:

$$D(t) = E(t) - I(t). \tag{15}$$

For $t < \hat{t} + R_1$ and $t > \hat{t} + \omega$ it holds that $D(t) = 0$. The present value of the deficit in the intermediate periods is given by:

$$\overline{D} = \int_{\hat{t}+R_1}^{\hat{t}+\omega} D(t) e^{-r(t-(\hat{t}+R_1))} \mathrm{d}t. \tag{16}$$

The budget-neutral value of x is given by the value for which $\overline{D}$ in equation (16) is equal to zero. This value will therefore depend on the market interest rate r. It can be calculated that for $r = 0$ and $x = 0$ it holds that:

$$\overline{D} = \frac{1}{2} \tau W N (R_1 - R_2)(\omega - A). \tag{17}$$

This means that for an upward jump in the retirement age ($R_2 > R_1$) even for $r = 0$ the system will not return to balance if $x = 0$. The pure NDC system will thus not me able to stabilize the budget in this case. One can derive an approximated expression for $\overline{D}$ and solve this for x to derive an approximated solution for the budget neutral deduction rate. It comes out as:

$$x^* = \frac{(\omega - A)(r(\omega + R_2 - 2R_1) - 3)}{3(R_2 - A)(\omega - R_2)}. \tag{18}$$

This can be compared to the standard value for budget neutral deductions x^s that is implied by the deduction equation (1) and setting $\delta = r$. Using the fact that for a NDC system one can write $X = 1 + x(R^* - R)$ and $\Delta = 1 + \frac{r}{2}(R - R^*)\frac{\omega - A}{R - A}$ (see table 1) one can derive that:

$$x^s = \frac{r}{2} \frac{\omega - A}{R_2 - A}. \tag{19}$$

This implies that even for $r = 0$ the budget neutral deduction rate is not $x = 0$ (as suggested by (19)) but rather $x^* = \frac{\omega - A}{(R_2 - A)(\omega - R_2)}$ as given by (18). For $A = 20$, $\omega = 80$ and a shift from $R_1 = 65$ to $R_2 = 70$ implies for example that $x^* = -0.094$ (exact value). The deduction rate had to be almost 10% in order to balance the budget over the shock interval. The level of r has not much effect on this magnitude. For $r = 0.02$ it is, e.g., given by $x^* = -0.101$. This is much lager than the standard deduction given by $x^s = -0.012$.

A.2 Two-point distribution (section 3.4)

It is now assumed that up to cohort $\hat{t}$ there are two possible retirement ages—a low age R_1^L and a high age $R_1^H \geq R_1^L$ that are chosen with relative frequencies p_1 and $(1 - p_1)$, respectively. For every cohort born before $\hat{t}$ the average (cohort) retirement age is thus given by $\overline{R}_1 = p_1 R_1^L + (1 - p_1) R_1^H$. From cohort $\hat{t}$ on, however, retirement behavior shifts and a fraction p_2 chooses an age $R_2^L > R_1^L$ while a fraction $(1 - p_2)$ chooses the age R_2^H (with $R_2^L \leq R_2^H \leq R_1^H$. The new average retirement age per cohort is now given by $\overline{R}_2 = p_2 R_2^L + (1 - p_2) R_2^H$. The shock thus assume that the retirement age distribution gets more narrow. The early retirees stay longer in the labor market while the late retirees leave earlier.

Before the shock occurs a NDC pension system is in continuous balance and the same is true for the periods after the shock has been digested if $\overline{R}_2 = \overline{R}_1 = R^*$. This follows from the fact that these are stationary distributions and proposition 1 states that in this case a NDC system without additional deductions will lead to a balanced budget. In the intermediate interval, however, there will be periods of budget surpluses and deficits. In order to discuss this formally one has to define the size of the retired population $M(t)$ and total expenditures $E(t)$. For the definitions one has to distinguish six intervals. Before time $\hat{t} + R_1^L$ one is in the old steady state world while after period $\hat{t} + \omega$ the new steady state is valid. The in-between periods, however, differ from these reference values. For the sake of simplicity I assume that $p_1 = p_2 = p$ and the size of each cohort is constant and denoted by N.

$$M(t)=\begin{cases} N(\omega-\overline{R}_1) & \text{for } t\le \hat{t}+R_1^L,\\ N(\omega-(1-p)R_1^H-p(t-\hat{t})) & \text{for } \hat{t}+R_1^L\le t\le \hat{t}+R_2^L,\\ N(\omega-pR_2^L-(1-p)R_1^H) & \text{for } \hat{t}+R_2^L\le t\le \hat{t}+R_2^H,\\ \left.\begin{array}{l} N\Big(p(\omega-R_2^L)+(1-p)((t-\hat{t})-R_2^H)+\\ (1-p)(\omega-R_1^H)\Big)\end{array}\right\} & \text{for } \hat{t}+R_2^H\le t\le \hat{t}+R_1^H,\\ N(\omega-\overline{R}_2) & \text{for } \hat{t}+R_1^H\le t\le \hat{t}+\omega,\\ N(\omega-\overline{R}_2) & \text{for } t\ge \hat{t}+\omega. \end{cases}$$

$$E(t)=\begin{cases} N\Big(p(\omega-R_1^L)P_1^L+(1-p)(\omega-R_1^H)P_1^H\Big) & \text{for } t\le \hat{t}+R_1^L,\\ N\Big(p(\omega-(t-\hat{t}))P_1^L+(1-p)(\omega-R_1^H)P_1^H\Big) & \text{for } \hat{t}+R_1^L\le t\le \hat{t}+R_2^L,\\ \left.\begin{array}{l} N\Big(p(\omega-(t-\hat{t}))P_1^L+p((t-\hat{t})-R_2^L)P_2^L+\\ (1-p)(\omega-R_1^H)P_2^H\Big)\end{array}\right\} & \text{for } \hat{t}+R_2^L\le t\le \hat{t}+R_2^H,\\ \left.\begin{array}{l} N\Big(p(\omega-(t-\hat{t}))P_1^L+p((t-\hat{t})-R_2^L)P_2^L+\\ (1-p)((t-\hat{t})-R_2^H)P_2^H+(1-p)(\omega-R_1^H)P_1^H\Big)\end{array}\right\} & \text{for } \hat{t}+R_2^H\le t\le \hat{t}+R_1^H,\\ \left.\begin{array}{l} N\Big(p(\omega-(t-\hat{t}))P_1^L+p((t-\hat{t})-R_2^L)P_2^L+\\ (1-p)((t-\hat{t})-R_2^H)P_2^H+\\ (1-p)(\omega-(t-\hat{t}))P_1^H\Big)\end{array}\right\} & \text{for } \hat{t}+R_1^H\le t\le \hat{t}+\omega,\\ N\Big(p(\omega-R_2^L)P_2^L+(1-p)(\omega-R_2^H)P_2^H\Big) & \text{for } t\ge \hat{t}+\omega. \end{cases}$$

The pension levels are given by:

$$P^j=\tau W\frac{R^j-A}{\omega-R^j}(1+x(R^*-R^j)), \tag{20}$$

where $j\in\{L1,L2,H1,H2\}$ and x is the deduction rate that is applied between periods $\hat{t}+R_1^L$ and $\hat{t}+\omega$.

The expressions for $L(t)$, $I(t)$ and $D(t)$ are still given by (13), (14) and (15). For $t<\hat{t}+R_1^L$ and $t>\hat{t}+\omega$ it holds that $D(t)=0$. The present value of the deficit in the intermediate periods is given by:

$$\overline{D} = \int_{\hat{t}+R_1^L}^{\hat{t}+\omega} D(t)e^{-r(t-(\hat{t}+R_1^L))}\mathrm{d}t. \tag{21}$$

The budget-neutral value of x is given by the value for which $\overline{D}$ in equation (21) is equal to zero. This value will therefore depend on the market interest rate r. It can be calculated that for $r = 0$ and $x = 0$ it holds that:

$$\overline{D} = \frac{1}{2}\tau W N(\overline{R}_1 - \overline{R}_2)(\omega - A),$$

which is completely analogous to equation (17) above. For the case where the average cohort retirement age is the same before and after the shock (i.e. $\overline{R}_2 = \overline{R}_1$) one has that $\overline{D} = 0$. In the case of zero market interest rate no additional deduction rates are necessary ($x = 0$) in order to guarantee an intertemporally balanced budget. In other words, in the intermediate interval the period deficits and period surpluses just counterbalance each other such that the total sum equals zero. This, however, is no longer true for $r > 0$. In this case the surpluses right after period $t = \hat{t} + R_1^L$ (the early retiree now postpone their retirement and thus there are higher contributions than normal) earn a positive interest rate and this can be used to reward late retirement (and punish late retirement).

I want to give a numerical example with with $\tau = 0.25$, $W = 100$ and $R_1^L = 60$, $R_1^H = 70$, $R_2^L = 65$, $R_2^H = 65$, $p_1 = p_2 = \frac{1}{2}$. Average cohort retirement ages are given by $\overline{R}_1 = \overline{R}_2 = 65$. For a market interest of $r = 0.02$ one can calculate that the budget-neutral deduction rate is given by $x = -0.0057$ while for $r = 0.05$ one gets $x = -0.014$. These values can be compared to the approach based on equation (1) where the deduction rates (using $\delta = r$) come out as $x = -0.0133$ (for $r = 0.02$) and $x = -0.0333$ (for $r = 0.05$). These "standard" deduction rates are thus considerably larger then the budget-neutral deduction rates. It can be calculated that their use would lead to a permanent surplus of 13.3% of revenues (for $r = 0.02$) and 210% (for $r = 0.05$), where the deficit ratio is calculated as $\overline{d} = \frac{\overline{D}}{\overline{I}}$. On the other hand, the waiving of additional deduction (i.e. the use of a pure NDC system with $x = 0$) would lead to permanent deficits in the magnitude of 10% of revenues (for $r = 0.02$) and 159% (for $r = 0.05$).

The two-point distribution with $\overline{R}_1 = \overline{R}_2$ is thus an interesting intermediate case. The budget-neutral deduction rate is given by 0.57% (for $r = 2\%$) which is in-between the standard value of 1.33% (that is budget-neutral for the one-time shock scenario) and the pure NDC value of 0% (that is budget-neutral for the case of a stationary retirement age distribution). The example is, however, rather stylized and primarily serves illustrative purposes. It requires, e.g., that the shock period and the future development is perfectly known (or anticiptaed) beforehand.

B The framework with fluctuating retirement (section 3.4): A discrete-time version of the model and the results of numerical simulations

B.1 Set-up

For the numerical simulations I have to use a discrete-time version of the model. I focus here on the simple model with rectangular mortality and zero growth. The cohort born in time t has $N(t)$ members. Each individual $i \in [0, N(t)]$ starts to work after the same age A, is continuously employed and dies at the same age ω. Individuals, however, might differ in the length of their working life. $C_i(t)$ denotes the number of years a person contributes to the pension system, while $R_i(t)$ is the age when the individual enters retirement. I assume that each individual has at least one period in the labor market and one period in retirement, i.e. $1 \leq C_i(t) < \omega$. It holds that $R_i(t) = A + C_i(t) + 1$. To give an example, assume that $A = 20$ and $C_i(t) = 45$. In this case the individual works between his 20th and 21st, 21st and 22nd etc. birthdays up to the period between his 64th and 65th birthday, a total of 45 period. The first retirement period is then between his 65th and 66th birthday or $R_i(t) = 66$.

One can define a variable $I_i^W(a, t)$ that indicates whether individual i, born in period t is working (and paying contributions) at the adult age $a \equiv \tilde{a} - A$ (where $\tilde{a}$ stands for the biological age). In particular, $I_i^W(a, t) = 1$ for $1 \leq a \leq C_i(t)$ and $I_i^W(a, t) = 0$ elsewhere. The density function of contribution years is then given by:

$$f(a, t) = \frac{\sum_{i=1}^{N(t)} I_i^W(a, t)}{N(t)} \text{ for } 1 \leq a < \omega. \tag{22}$$

One could also specify a density function of retirement ages that is defined in an analogous manner to (22), but I prefer the specification based on contribution years since it is more directly related to the calculation of pension benefits.

For an individual with contribution years $C_i(t) = C$ and target contribution years $C^* > C$ the discrete-time equivalent to the continuous-time deduction equation (1) can be written as:

$$\sum_{a=C+1}^{C^*}\Big(\tau W+\widehat{P}\chi(C,t)\Big)\frac{1}{(1+\delta)^{a-(C+1)}}=\sum_{a=C^*+1}^{\omega}\Big(P^*-\widehat{P}\chi(C,t)\Big)\frac{1}{(1+\delta)^{a-(C+1)}},\quad (23)$$

where for the NDC system $\widehat{P}=\frac{C}{\omega-C}$ and $P^*=\frac{C^*}{\omega-C^*}$. I have written here $\chi(C,t)$ for the deductions in order to indicate that they might depend on the specific time period and on the number of contribution years (or the retirement age). The values for $\chi(C,t)$ based on this discrete-time expression (23) are close to the ones reported in tables 2 and 3 for the continuous-time framework. In order to derive deduction factors that are able to stabilize the budget in the long-run I look at an intertemporal balanced budget condition like in equation (10) for the continuous-time framework. In order to do so it is helpful to first define the period deficit as:

$$\begin{aligned}D(t) &= \sum_{C=1}^{\omega-1}\sum_{a=C+1}^{\omega}\left(\tau W\frac{C}{\omega-C}(1+x(C^*-C))f(C,t-a+1)\right)-\\ &\quad \tau W\left(\omega-\sum_{C=1}^{\omega-1}\sum_{a=C+1}^{\omega}f(C,t-a+1)\right),\end{aligned}\quad (24)$$

where the first term in (24) refers to expenditures and the second equals revenues. The intertemporal balanced budget condition can then be expressed as:

$$\sum_{t=t_0}^{t_T}D(t)\frac{1}{(1+r)^{t-t_0}}=0.\quad (25)$$

B.2 Simulation runs

In the paper I sketch the results of one particular numerical simulation. I have also performed simulations for different assumptions that have lead to similar results (that are not reported in the following).

In the simulation I start from a target distribution of contribution years which is (symmetrically) triangular between 40 and 50 with a mean at $\overline{C}^*=45$ that is also assumed to represent the target contribution years C^*. In the simulations the contribution years for the individuals are set in such a manner that the target distribution resembles the triangular benchmark as closely as possible. The simulations start in a stationary situation (where the actual distribution is identical to the target distribution). From a specific cohort $\hat{t}$ onward, however, the actual distribution is given by random draws from the target distribution until at the end the actual distribution again returns to the target. This benchmark simulation scenario is repeated 100 times.

The parameters of the simulations are chosen in such a way that the retirement fluctuations during the shock periods roughly correspond to realistic values. In particular, I only use 100 individuals for each simulation run. The larger the sample size the more the actual distributions coincide with the target distribution thereby returning to the case of a stationary distribution. For the one-hundred simulations runs I get the following summary statistics: The target distribution has (per assumption) mean contribution years of 45 with a standard deviation of 2.04.[18] Since the actual distributions are random draws from this target, one would expect that the average values over the shock periods correspond to the summary statistics of the target distribution. This is in fact the case in the simulated data where the average contribution years (over all shock periods and all simulation runs) is exactly 45 with an average standard deviation of 2.05. The average standard deviation over these shock periods is 0.18 for the mean of the contribution years and 0.11 for the standard deviation.

These values can be compared to real-world data. In particular, I use data from Austria from the years 2005 to 2011 (see Statistik Austria 2013). In figure A.1 I illustrate the retirement probabilities for males in these 7 years together with their means. The shape of the distribution loosely resembles a triangle. One can observe that around one third of all individuals enter retirement at the age of 60 while there is an additional spike at age 57 and a small one at the statutory retirement age 65. This highlights the prevalence of early retirement in Austria (at least for this time period). This, however, is not the topic of the present paper. For the question of accurate deductions it is more interesting to note the relative stability of the actual distributions of retirement ages around the mean values (from 2005 to 2011) which could be assumed to correspond to the target distribution. The average retirement age over these years is given by 59.5 with a standard deviation (from 2005 to 2011) of 0.14. The average standard deviation of the retirement ages (between 50 and 70) comes out as 2.98 with a standard deviation (from 2005 to 2011) of 0.053. These values are broadly comparable to the ones used in the simulation above.[19]

For each of the 100 simulations runs I calculate x. The average value over the 100

[18] Note that for a continuous triangular distribution the standard deviation is given by $5/\sqrt{6} = 2.04$.

[19] Note that the simulation is formulated in terms of the contribution years while the data refer to retirement ages. The two are, however, closely connected and the standard deviations can be expected to be similar. I do not try to match the moments of the empirical data precisely since I am mainly interested in qualitative results.

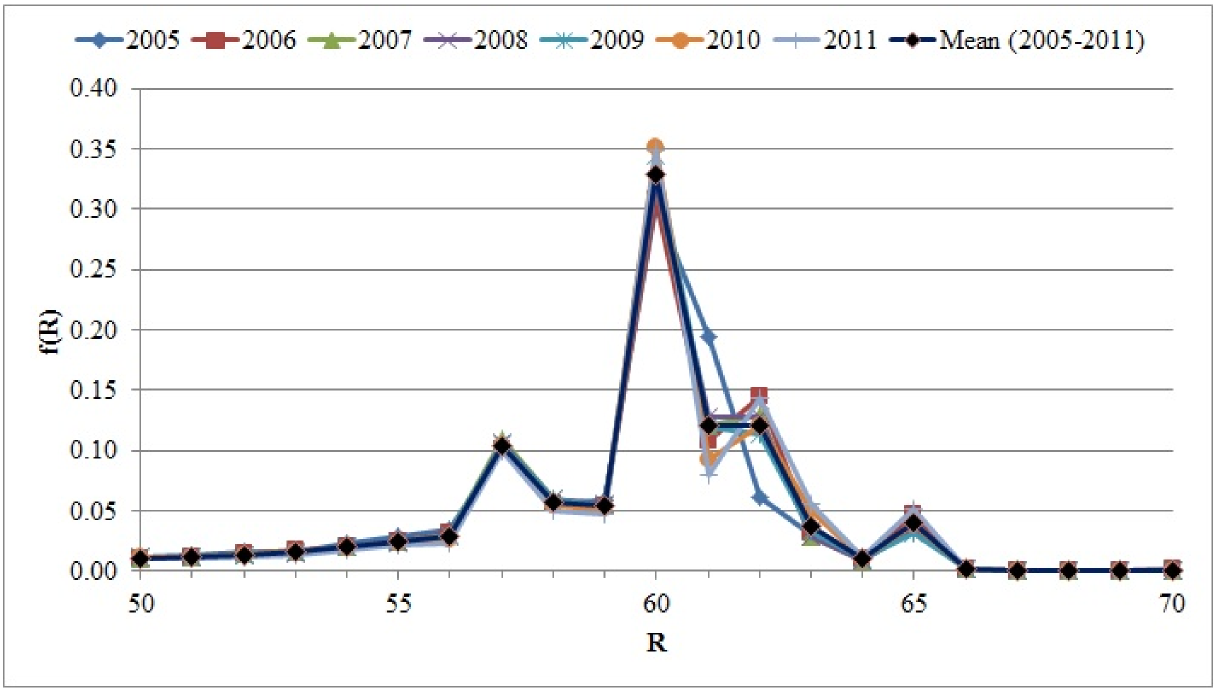
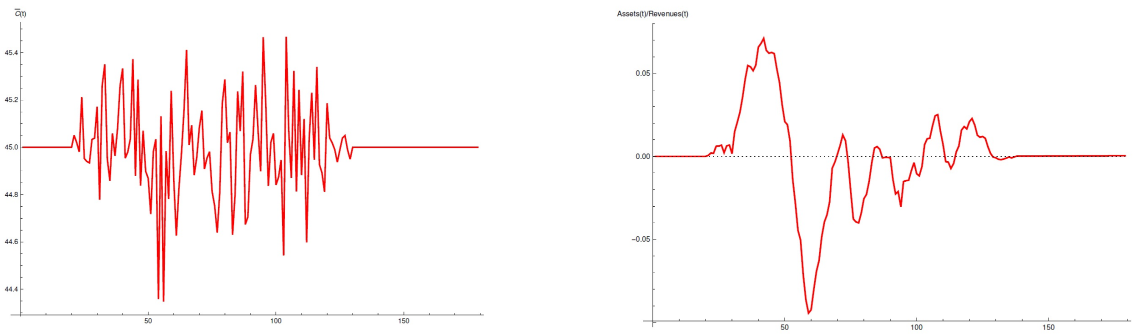


Figure A.1: The picture show the retirement probability for Austrian males in the years 2005 to 2011 (i.e. the values $f(R, 2005 - R)$, $f(R, 2006 - R)$, ..., $f(R, 2011 - R)$). The average retirement age over the years is given by 59.5 with a standard deviation (from 2005 to 2011) of 0.14. The standard deviation of the retirement ages (between 50 and 70) comes out as 2.98 with a standard deviation (from 2005 to 2011) of by 0.053.

simulations runs comes out as $\overline{x} = 0.0002$ which is very close to the value of stationarity (where one has no extra deductions, i.e. $x = 0$). In figure A.2 I illustrate the fluctuations in the average contribution years together with the budgetary implications for one specific simulation (that is associated with a deduction rate x that is close to 0). In particular, I show the development of the assets-to-revenues level of the pension system (i.e. the negative of the debt-to-revenues stock) for the use of a NDC system together with a deduction rate of $x = -0.0000325$. The interest rate is assumed to be given by $r = 0.02$. As can be seen, the asset stock fluctuates quite a bit even though average contribution years $\overline{C}(t)$ only fluctuate between 44.4 and 45.5. In the end, however, the asset level returns to zero and the pension system is again in balance. If the pension system had instead used deductions that are based on the benchmark approach (i.e. given by values around -0.0145, see tables 2 and 3) then the picture is different. The simulations indicate (not shown) that in this case the pension system does not return to balance but rather shows a permanent (and exploding) surplus.



(a) Average Contribution Years (b) Assets-Revenues-Ratio

Figure A.2: Panel (a) shows the fluctuations of the average contribution years for one specific simulation where from some cohort onwards the contribution years are random draws from a triangular distribution. Panel (b) shows the implied fluctuations in the asset-to-revenues-ratio if pensions are based on a NDC system with a deduction rate of $x = -0.0000325$ and the interest rate is given by $r = 0.02$.

C Stationary retirement in a model with growing wages and non-rectangular mortality

This generalizes the model of section 3. The main results of the generalized model are summarized in section 5.1 of the paper.

C.1 Set-up

C.1.1 Demographic structure

I work with a model in continuous time. In every instant of time t a new cohort is born. The maximum age that a member of cohort t can reach is time-invariant and denoted by ω. $S(a)$ gives the probability that an individual survives to age a. It holds that $S(0) = 1$, $S(\omega) = 0$ and that survivorship declines with age, i.e. $\frac{dS(a)}{da} \leq 0$ for $a \in [0, \omega]$. The mortality hazard rate is given by $\mu(a) \equiv -\frac{dS(a)}{da}\frac{1}{S(a)}$. Therefore:

$$S(a) = e^{\int_0^a -\mu(x)\,\mathrm{d}x}. \tag{26}$$

An interesting benchmark case is given by rectangular survivorship where $S(a) = 1$ for $a \in [0, \omega]$. In this case there are no premature deaths and all members of a cohort reach the maximum age ω. This corresponds to the assumption made in section 3.

Remaining life expectancy is given by:[20]

$$e(z) = \int_z^\omega e^{\int_z^a -\mu(x)\,\mathrm{d}x}\,\mathrm{d}a = \frac{\int_z^\omega S(a)\,\mathrm{d}a}{S(z)}. \tag{27}$$

The second equality follows from the fact that $e^{\int_z^a -\mu(x)\,\mathrm{d}x} = e^{\int_0^a -\mu(x)\,\mathrm{d}x} e^{\int_0^z \mu(x)\,\mathrm{d}x} = \frac{S(a)}{S(z)}$ where the last step uses equation (26).

The size of cohort t at age a is given by $N(a,t) = N(0,t)S(a)$, where $N(0,t)$ stands for the initial size of the cohort. For the sake of simplicity I assume constant sizes of birth cohorts, i.e. $N(0,t) = N, \forall t$. The entry age in the labor market is again assumed to be constant and given by A while the age-specific probability to retire for generation t is again given by $f(a,t)$ for $a \in [A,\omega]$. I assume that the mortality rates are independent from this probability. The cumulative function $F(a,t)$ then gives the percentage of the surviving members of cohort t that are already retired at age a. It holds that $F(A,t) = 0$ and $F(\omega,t) = 1$. In this appendix I focus on a stationary retirement distribution, i.e. $f(a,t) = f(a)$ and $F(a,t) = F(a)$.

The total size of the active population L and the retired population M are constant and given by:

$$L = N \int_A^\omega S(a)\,(1 - F(a))\,\mathrm{d}a, \tag{28}$$

$$M = N \int_A^\omega S(a)F(a)\,\mathrm{d}a. \tag{29}$$

C.1.2 Budget of the pension system

The contribution rate to the PAYG pension system is assumed to be fixed at τ. I abstract from intragenerational wage differences and seniority profiles and simply assume that in a specific period t each workers earns an identical wage $W(t)$. Wages grow at rate $g(t)$, i.e. $W(t) = W(0)e^{\int_0^t g(s)\,\mathrm{d}s}$.

Each retired member of generation t receives a pension payment $P(R,a,t)$. The size of the pension can depend on the payment period $t+a$, on the individual's age a and also on the time of his or her retirement $R \le a$. Below I will say more about the determination of the pension payments in different systems.

In order to calculate the total expenditures of the pension system one can make the following considerations. First, focus on one particular retirement age R and calculate the total of pension payments that is distributed to the group of pensioners that has

[20] See, e.g., Keyfitz & Caswell (2005) or Goldstein (2006).

retired at this age. This comprises individuals at different ages $a \in [R, \omega]$. For a person who is of age a in period t the pension payment is $P(R, a, t-a)$ and the size of this subgroup is $N \times S(a)$. The total payments to people with retirement age R in period t is thus given by: $P^{total}(R,t) = N \int_R^\omega P(R,a,t-a)S(a)\,\mathrm{d}a$. The same logic applies for any possible retirement age $R \in [A, \omega]$ where the relative frequency of the retirement age is given by $f(R)$. Total pension expenditures in period t can thus be written as $E(t) = \int_A^\omega P^{total}(R,t) f(R)\,\mathrm{d}R$ or:[21]

$$E(t) = N \int_A^\omega \left(\int_R^\omega P(R,a,t-a)S(a)\,\mathrm{d}a \right) f(R)\,\mathrm{d}R. \tag{30}$$

Total revenues $I(t)$, on the other hand, are given by:

$$I(t) = \tau W(t) L = \tau W(t) N \int_A^\omega S(a)\left(1 - F(a)\right)\,\mathrm{d}a. \tag{31}$$

The total deficit (or surplus) of the pension system is given by $D(t) = E(t) - I(t)$ while the deficit ratio by $d(t) = \frac{D(t)}{I(t)} = \frac{E(t)}{I(t)} - 1$. A continuously balanced budget is thus characterized by $D(t) = d(t) = 0, \forall t$.

C.2 Different PAYG systems

In the last section C.5 of this appendix I discuss in detail how the pension level $P_j(R,a,t)$ is determined in the three different pension systems $j \in \{\mathrm{DB}, \mathrm{AR}, \mathrm{NDC}\}$. Here I only summarize the main results. There are two main differences to the simple model of sections 2 and 3 that have to do with the assumptions of growth and mortality. All pension systems have to specify how pension claims that have been acquired in the past are revalued at the moment of retirement. In the NDC system, e.g., this is done by the choice of a "notional interest rate" $\rho(a,t)$. There exist two popular variants of this interest rate that are discussed in the literature and used in real-world systems:

$$\rho(a,t) = g(t) + \mu(a) \tag{32a}$$

or

$$\rho(a,t) = g(t). \tag{32b}$$

[21] Alternatively, one could also reverse the order and look first at a fixed age a of the retired population and calculate the total pension payments to those individuals that have different retirement ages R. In the second step one would then calculate the total pension payments to all possible ages a. This results in an equivalent expression. I focus on the formulation in (30) since it is more convenient for later calculations.

Both notional interest rates reflect the growth rate of average wages $g(t)$ while the first specification (32a) also corrects for the fact that each period some cohort members die. The account values of the deceased cohort members are regarded as "inheritance gains" that are distributed among surviving cohort members by granting an extra return $\mu(a)$.

For later reference it is also useful to define the following term:

$$h(R) \equiv \frac{\int_A^R S(a)\,\mathrm{d}a}{(R-A)S(R)}, \tag{33}$$

that stands for the "per capita inheritance gains premium", i.e. the factor by which the first pension at retirement age R is higher if the revaluation takes inheritance gains into account. For rectangular survivorship there are no inheritance gains and thus the average premium is $h(R) = 1$.

Furthermore, due to the assumption of a growing economy one also has to specify how pension are adjusted over time. Here it is assumed (for simplicity and in line with the practice in many countries) that ongoing pensions are adjusted with the average growth rate of wages, i.e.%

$$\vartheta(t) = g(t). \tag{34}$$

In section 5.3 of the paper I also discuss briefly the consequences of the assumption $\vartheta(t) = 0$.

In section C.5 I show that in this situation the three pension system are associated with the following pension levels:

$$P_{\mathrm{NDC}}(R, a, t) = \tau W(t+a)\frac{(R-A)h(R)}{e(R)}X_{\mathrm{NDC}}(R, t), \tag{35}$$

$$P_{\mathrm{DB}}(R, a, t) = q^{*}W(t+a)X_{\mathrm{DB}}(R, t), \tag{36}$$

$$P_{\mathrm{AR}}(R, a, t) = \kappa^{*}W(t+a)(R-A)X_{\mathrm{AR}}(R, t), \tag{37}$$

where the NDC system uses the notional interest including inheritance gains (i.e. equation (32a)) to revalue past contributions. This is in line with the approach used in Sweden. The other two system are based on indexations excluding these mortality adjustments (i.e. on equation (32b)) which is also in line with the real-world systems. All three pension system also allow for the use of demographic adjustment factors $X_j(R, t)$ as discussed in section 2. In the case of non-stationary constellations the deduction factor might also be time-dependent as is indicated by the use of a time index t.

C.3 Budget-neutral deductions

In this part I investigate how the deduction factors $X_j(R,t)$ have to be determined in order to guarantee a balanced PAYG system in the case of a stationary demographic situation. The following proposition is a generalization of proposition 1. It states that a standard NDC system with $\rho(a,t) = g(t) + \mu(a)$ leads to a balanced budget without the need for further deductions.

Proposition 2

Assume a stationary demographic situation where the size of birth cohorts is constant ($N(0,t) = N$), people start to work at age A, the maximum age is ω, mortality is described by the survivorship function $S(a)$ for $a \in [0,\omega]$, retirement age is distributed according to the density function $f(R)$ for $R \in [A,\omega]$ and wages grow with rate $g(t)$. In this case a NDC system will be in continuous balance ($E(t) = I(t), \forall t$) if the notional interest rate and the adjustment factor are set according to $\rho(a,t) = g(t) + \mu(a)$ (equation (32a)) and $\vartheta(t) = g(t)$ (equation (34)), respectively, and if there are no additional deductions ($X_{NDC}(R,t) = 1$).

Proof. For the NDC system one can insert the pension level from equation (35), i.e. $P_{\text{NDC}}(R,a,t-a) = \tau W(t)\frac{(R-A)h(R)}{e(R)}X_{\text{NDC}}(R,t)$ into (30) to conclude that:

$$
\begin{aligned}
E(t) &= \tau W(t) N \int_A^\omega \frac{(R-A)h(R)}{e(R)} X_{\text{NDC}}(R,t) \left(\int_R^\omega S(a)\,\mathrm{d}a \right) f(R)\,\mathrm{d}R \\
&= \tau W(t) N \int_A^\omega \left(\int_A^R S(a)\,\mathrm{d}a \right) X_{\text{NDC}}(R,t) f(R)\,\mathrm{d}R, \qquad (38)
\end{aligned}
$$

where I use the definitions $h(R) = \frac{\int_A^R S(a)\,\mathrm{d}a}{(R-A)S(R)}$ and $e(R) = \frac{\int_R^\omega S(a)\,\mathrm{d}a}{S(R)}$.

For the assumptions of proposition 2 total expenditures in equation (38) can be written as:

$$
E(t) = \tau W(t) N \int_A^\omega \left(\int_A^R S(a)\,\mathrm{d}a \right) f(R)\,\mathrm{d}R.
$$

One can define $u(R) = \int_A^R S(a)\,\mathrm{d}a$ and $v(R) = 1 - F(R)$ with $u'(R) = S(R)$ and $v'(R) = -f(R)$. Using integration by parts it holds that:

$$
\int_A^\omega \left(\int_A^R S(a)\,\mathrm{d}a \right) f(R)\,\mathrm{d}R = -\int_A^\omega u(R)v'(R)\,\mathrm{d}R = -u(R)v(R) + \int_A^\omega u'(R)v(R)\,\mathrm{d}R.
$$

The term $(-u(R)v(R))$ is given by $\left[\left(\int_A^R S(a)\,\mathrm{d}a\right)(1-F(R))\right]_A^\omega$ which can be evaluated as

$$\left(\int_A^\omega S(a)\,\mathrm{d}a\right)(1-F(\omega)) - \left(\int_A^A S(a)\,\mathrm{d}a\right)(1-F(A)) = 0. \tag{39}$$

Since it holds that $\int_A^\omega u(R)v'(R)\,\mathrm{d}R = \int_A^\omega S(R)(1-F(R))\,\mathrm{d}R$ one can conclude that:

$$E(t) = \tau W(t) N \int_A^\omega S(R)(1-F(R))\,\mathrm{d}R. \tag{40}$$

This is equal to total revenues $I(t) = \tau W(t) N \int_A^\omega S(a)\,(1-F(a))\,\mathrm{d}a$ (see (31)) and thus $E(t) = I(t)$. ■

Proposition 2 generalizes proposition 1 and it confirms the previous findings. For a stationary economic and demographic situation a NDC system that includes a correction for inheritance gains is stable if one uses the benchmark NDC formula:

$$P_{\mathrm{NDC}}(R,a,t) = \tau W(t+a)\frac{(R-A)h(R)}{e(R)}. \tag{41}$$

There is no need for an additional adjustment factor and it holds (as in section 3.2) that $X_{\mathrm{NDC}}(R,t) = \Psi_{\mathrm{NDC}}(R) = 1$.

In this case it is also possible to make the DB and the AR systems stable by just using demographic deduction factors $\Psi_{\mathrm{DB}}(R)$ and $\Psi_{\mathrm{AR}}(R)$ that are independent of the discount rate δ and of time t. These are calculated in appendix C.5.[22] As in section 2 I again invoke the "balanced target condition", i.e. the condition that the target replacement rate q^* is chosen in such a way that if everybody retires at the target retirement age R^* there will be no deductions ($\Psi_{\mathrm{DB}}(R^*) = 1$) and the system will be in balance (see section 5.3 for a discussion of the situation without a balanced budget). The demographic adjustment factors come out as:

$$\Psi_{\mathrm{DB}}(R) = \frac{e(R^*)}{e(R)}\frac{h(R)}{h(R^*)}\frac{R-A}{R^*-A}, \tag{42}$$

$$\Psi_{\mathrm{AR}}(R) = \frac{e(R^*)}{e(R)}\frac{h(R)}{h(R^*)}. \tag{43}$$

In table A.1 I collect important formulas for the three PAYG systems. In particular, it contains the formula pension $\widehat{P}_j(R,a,t)$ (both in its basic form and after invoking the

[22]I also discuss the case of a NDC system that uses (32b) instead of (32a), i.e. that excludes the correction for inheritance gains in the notional interest rate. In this case one also needs a demographic adjustment factor in order to implement a balanced system.

Table A.1: Three PAYG systems for a stationary demography

Type (j)	(1) $\widehat{P}_j(R,a,t)$	(2) $\widehat{P}_j(R,a,t)$ (for BTC)	(3) Ψ_j	(4) $P_j(R,a,t)$ (for BTC)
DB	$q^* W(t+a)$	$\tau W(t+a)\frac{(R^*-A)h(R^*)}{e(R^*)}$	$\frac{e(R^*)}{e(R)}\frac{h(R)}{h(R^*)}\frac{R-A}{R^*-A}$	$\tau W(t+a)\frac{(R-A)h(R)}{e(R)}$
AR	$\kappa^*(R-A)W(t+a)$	$\tau W(t+a)\frac{(R-A)h(R^*)}{e(R^*)}$	$\frac{e(R^*)}{e(R)}\frac{h(R)}{h(R^*)}$	$\tau W(t+a)\frac{(R-A)h(R)}{e(R)}$
NDC	$\tau W(t+a)\frac{(R-A)h(R)}{e(R)}$	$\tau W(t+a)\frac{(R-A)h(R)}{e(R)}$	1	$\tau W(t+a)\frac{(R-A)h(R)}{e(R)}$

Note: The table shows the formula pension $\widehat{P}_j(R,a,t)$, the demographic deduction factor Ψ_j and the total pension $P_j(R,a,t) = \widehat{P}_j(R,a,t)\Psi_j$ for three variants of PAYG schemes: DB (Defined Benefit), AR (Accrual Rates), NDC (Notional Defined Contribution). The balanced target condition (BTC) has to hold if the system has a balanced budget in the case that all individuals retire at the target retirement age $R = R^*$. These are specified in the text. Column (4) is the multiple of columns (2) and (3).

balanced target condition), the demographic deduction factor Ψ_j and the final pension $P_j(R,a,t) = \widehat{P}_j(R,a,t)\Psi_j$. Note that for rectangular survivorship it holds that $h(R) = 1$ and $e(R) = \omega - R$. In this case the results of table A.1 coincide with the ones of table 1. In particular, $\Psi_{\mathrm{DB}}(R) = \frac{\omega-R^*}{\omega-R}\frac{R-A}{R^*-A}$ and $\Psi_{\mathrm{AR}}(R) = \frac{\omega-R^*}{\omega-R}$.

C.4 The choice of discount rates

So far I have shown that for a stationary economic and demographic situation a standard NDC system implements a stable PAYG pension system. By using the correct demographic adjustment factors Ψ_{DB} and Ψ_{AR} also the DB and a AR systems can be amended to guarantee a continuous budgetary balance. This implies that it is not necessary to refer to the market interest rate in order to design the budget-neutral deduction rates in this stationary constellation.

It is interesting to look at this issue from the viewpoint of the standard deduction framework presented in section 2.2 and ask a number of questions. First, which choice of the discount rate will give rise to the budget-neutral demographic deduction factors Ψ_j? Second, what deductions are implied if the discount rate is set to higher levels? Third, under which conditions will higher discount rates also be compatible with a balanced budget? Answers to these questions will be provided in the next three subsections.

C.4.1 The appropriate budget-neutral discount rate for a stationary demography

In order to find the discount rate that is compatible with the budget-neutral demographic deduction factors Ψ_j one has to adapt the neutrality condition (1) of section 2.2 for the general framework. It comes out as:

$$\begin{aligned}\textstyle\int_R^{R^*} \left(\tau W(t+a) + \widehat{P}_j(R,a,t)X_j\right) e^{-\delta(a-R)} S(a)\,\mathrm{d}a =\\ \textstyle\int_{R^*}^{\omega} \left(\widehat{P}_j(R^*,a,t) - \widehat{P}_j(R,a,t)X_j\right) e^{-\delta(a-R)} S(a)\,\mathrm{d}a.\end{aligned} \tag{44}$$

I want to know for which choice of δ the total deduction factor will collapse to the demographic factor, i.e. for which δ it holds that $X_j = \Psi_j$. In general, one cannot solve (44) for X_j in closed form. In the following I show, however, analytically that for the case of constant growth (i.e. $g(t) = g$) the choice of $\delta = g$ leads to the result that $X_j = \Psi_j$.

In order to do so I focus on formula pensions that are proportional to $\tau W(t+a)$. Therefore I write $\widehat{P}(R,a,t) = \tau W(t+a)\check{P}(R)$. Furthermore, noting that $W(t+a) = W(t+R)e^{\int_R^a g(t+s)\,\mathrm{d}s}$ equation (44) can also be written as:

$$\begin{aligned}\tau W(t+R)\int_R^{R^*} \left(1+\check{P}(R)X\right) e^{\int_R^a g(t+s)\,\mathrm{d}s} e^{-\delta(a-R)} S(a)\,\mathrm{d}a =\\ \tau W(t+R)\int_{R^*}^{\omega} \left(\check{P}(R^*) - \check{P}(R)X\right) e^{\int_R^a g(t+s)\,\mathrm{d}s} e^{-\delta(a-R)} S(a)\,\mathrm{d}a.\end{aligned}$$

For constant wage growth $g(s) = g$ this can be simplified to:[23]

$$\begin{aligned}\textstyle\int_R^{R^*} \left(1+\check{P}(R)X\right) e^{-(\delta-g)(a-R)} S(a)\,\mathrm{d}a =\\ \textstyle\int_{R^*}^{\omega} \left(\check{P}(R^*) - \check{P}(R)X\right) e^{-(\delta-g)(a-R)} S(a)\,\mathrm{d}a.\end{aligned} \tag{45}$$

I want to show that for the choice of $\delta = g$ the deductions X coincide with the demographic deduction factor Ψ. I focus first on the NDC system. In this case it holds that $\Psi = 1$ and the conjecture is that $X = \Psi = 1$. Furthermore, $\widehat{P}(R,a,t) = \tau W(t+a)\frac{(R-A)h(R)}{e(R)}$, i.e. $\check{P}(R) = \frac{(R-A)h(R)}{e(R)}$. Noting that $h(R) = \frac{\int_A^R S(a)\,\mathrm{d}a}{(R-A)S(R)}$ and $e(R) = \frac{\int_R^{\omega} S(a)\,\mathrm{d}a}{S(R)}$ one can thus

[23] Similarly, one could also assume a time-varying discount rate $\delta(s)$. Under the assumption that $\delta(s) = g(s)$ one could then derive the same result as in the following.

write: $\check{P}(R) = \frac{\int_A^R S(a)\,\mathrm{d}a}{\int_R^\omega S(a)\,\mathrm{d}a}$. Inserting this into (45) leads to:

$$\left(1+\frac{\int_A^R S(a)\,\mathrm{d}a}{\int_R^\omega S(a)\,\mathrm{d}a}\right)\int_R^{R^*} S(a)\,\mathrm{d}a = \left(\frac{\int_A^{R^*} S(a)\,\mathrm{d}a}{\int_{R^*}^\omega S(a)\,\mathrm{d}a} - \frac{\int_A^R S(a)\,\mathrm{d}a}{\int_R^\omega S(a)\,\mathrm{d}a}\right)\int_{R^*}^\omega S(a)\,\mathrm{d}a.$$

This can be simplified to:

$$\left(\int_R^\omega S(a)\,\mathrm{d}a + \int_A^R S(a)\,\mathrm{d}a\right)\int_R^{R^*} S(a)\,\mathrm{d}a =$$
$$\left(\int_A^{R^*} S(a)\,\mathrm{d}a \int_R^\omega S(a)\,\mathrm{d}a - \int_A^R S(a)\,\mathrm{d}a \int_{R^*}^\omega S(a)\,\mathrm{d}a\right).$$

Collecting terms this leads to:

$$\int_R^\omega S(a)\,\mathrm{d}a\left(\int_R^{R^*} S(a)\,\mathrm{d}a - \int_A^{R^*} S(a)\,\mathrm{d}a\right) = \int_A^R S(a)\,\mathrm{d}a\left(-\int_{R^*}^\omega S(a)\,\mathrm{d}a - \int_R^{R^*} S(a)\,\mathrm{d}a\right).$$

Combining integrals one can conclude:

$$-\left(\int_R^\omega S(a)\,\mathrm{d}a \int_A^R S(a)\,\mathrm{d}a\right) = -\left(\int_R^\omega S(a)\,\mathrm{d}a \int_A^R S(a)\,\mathrm{d}a\right).$$

This proves the conjecture that for $\delta = g$ the implied deduction X equals the demographic deduction Ψ which is just $\Psi = 1$ for the NDC system. Since the demographic deduction factors Ψ_{DB} and Ψ_{AR} are just determined in such a way as to transform the DB and AR systems into a NDC system the same conclusion also holds for these systems.

The stated result implies that the appropriate budget-neutral discount rate for a stationary situation is given by the internal rate of return. This has often been claimed in the related literature but the present framework allows to formulate it in a precise manner and to state the exact conditions (in particular demographic stationarity) under which it holds.

C.4.2 Deductions for different discount rates

In a next step one can investigate which deductions are implied by choices of the discount rate $\delta > g$. Although these choices are not necessary from the viewpoint of budgetary stability it is nevertheless instructive to see the magnitudes involved. To do so I use illustrative numerical examples. In particular, I assume a Gompertz survival curve of the

Table A.2: Deductions for $R = 64$ and $R^* = 65$

			$\hat{\delta} = 0$			$\hat{\delta} = 0.02$			$\hat{\delta} = 0.05$		
Type j	$\widehat{P}_j$	Ψ_j	X_j	x_j(in%)	P_j	X_j	x_j(in%)	P_j	X_j	x_j(in%)	P_j
DB	67.0	0.93	0.93	-7.04	62.3	0.91	-8.71	61.1	0.89	-11.49	59.3
AR	65.5	0.95	0.95	-4.92	62.3	0.93	-6.64	61.1	0.91	-9.48	59.3
NDC	62.3	1	1	0	62.3	0.98	-1.81	61.1	0.95	-4.79	59.3

Note: The table shows the actuarial deduction factors X_j, the annual deductions rates x_j (based on the linear relation $x_j = \frac{X_j - 1}{R^* - R}$) and the final pension $P_j(R, R^*) = \widehat{P}_j(R, R^*)X_j$ for three pension schemes and three (net) discount rates $\hat{\delta} \equiv \delta - g$. For the sake of comparison also the values of the pure demographic deduction factors Ψ_j are reported. The numerical values are: $A = 20$, $\tau = 0.25$, $g = 0.02$, $R^* = 65$ and $R = 64$. For all three schemes the target pension is $P^* = 67$. Mortality follows a Gompertz distribution with $\alpha = 0.000025$ and $\beta = 0.096$.

form $S(a) = e^{\frac{\alpha}{\beta}\left(1-e^{\beta a}\right)}$.[24]

In Tables A.2 and A.3 I report the deduction factors X_j and annual deduction rates x_j for various assumption concerning the (net) discount rate $\hat{\delta} \equiv \delta - g$. They correspond to tables 2 and 3 from section 2.2 which were based on rectangular survivorship. In addition, I also report the demographic deduction factors Ψ_j that are sufficient for budgetary stability. For all three systems the target pension level at the target retirement age $R^* = 65$ is given by $P^* = 67$ (which implies a target replacement rate for the DB system of $q^* = 0.67$). This is lower than for the case of rectangular survivorship since —due to premature deaths— remaining life expectancy $e(R^*)$ at $R^* = 65$ is now larger (18.6 vs. 15). The step-up of pensions due to inheritance gains is non-trivial and given by $h(R^*) = 1.11$ (or 11%). This is due to the fact that only 88% of all members of a cohort survive up to this age ($S(R^*) = 0.88$).

The results are qualitatively similar to the ones for rectangular survivorship in tables 2 and 3. The first thing to note is that for $\hat{\delta} = 0$ (i.e. $\delta = g$) the total deductions correspond exactly to the demographic adjustment factors, i.e. $X_j = \Psi_j$ as has already been shown in section C.4.1. For the DB system the correct annual deduction rate for a retirement at the age of 64 is 7% while it is 4.9% for the AR system if mortality follows a Gompertz

[24]The associated mortality rate is given by $\mu(a) = \alpha e^{\beta a}$ (i.e. the logarithm of mortality rates increases linearly in age). The Gompertz-function delivers a good description of empirical mortality data. For the following examples I use a parameterization of $\alpha = 0.000025$ and $\beta = 0.096$. This is roughly based on the Austrian life tables from 2005 which have the convenient property that the (unisex) life expectancy at birth was almost exactly 80 which facilitates comparisons to the numerical examples of section 2 with rectangular survivorship.

Table A.3: Deductions for $R = 60$ and $R^* = 65$

			$\hat{\delta} = 0$			$\hat{\delta} = 0.02$			$\hat{\delta} = 0.05$		
Type j	$\widehat{P}_j$	Ψ_j	X_j	x_j(in%)	P_j	X_j	x_j(in%)	P_j	X_j	x_j(in%)	P_j
DB	67.0	0.7	0.7	-5.94	47.1	0.64	-7.28	42.6	0.53	-9.36	35.6
AR	59.5	0.79	0.79	-4.18	47.1	0.72	-5.69	42.6	0.6	-8.04	35.6
NDC	47.1	1	1	0	47.1	0.9	-1.91	42.6	0.76	-4.87	35.6

Note: See table A.2 with the difference that now $R = 60$ instead of $R = 64$.

pattern. This is smaller than the corresponding rates for rectangular survival where they have been calculated as 8.33% and 6.25%, respectively. For larger discount rates, however, the difference shrinks and for $\hat{\delta} = 0.05$, e.g., the annual deduction rates for the Gompertz and the rectangular case are very similar (11.49%, 9.48% and 4.79% vs. 11.81%, 9.8% and 3.79%). A parallel conclusion holds for the case of $R = 60$ where the differences between the results of table A.3 (Gompertz) and table 3 (rectangular) are even smaller.

C.4.3 Balanced and unbalanced budgets for different discount rates

In the simple framework of sections 2 and 3 I have shown that even for discount rates that are larger than necessary it might still be the case that the system runs a balanced budget. In particular, I have shown in section 4 that the balanced budget condition depends on the choice of both the discount rate δ and the target retirement age R^*. This can be repeated for the general framework. It is not possible to derive closed-form solutions for the balanced budget and one has to resort to numerical calculations. In particular, I assume that retirement ages follow a triangular distribution that is defined by the minimum and maximum retirement ages R^L and R^H, respectively and also by the mode R^{mod}. The density function is then given by $f(R) = \frac{2(R-R^L)}{(R^H-R^L)(R^{mod}-R^L)}$ for $R^L \leq R \leq R^{mod}$ and $f(R) = \frac{2(R^H-R)}{(R^H-R^L)(R^H-R^{mod})}$ for $R^{mod} \leq R \leq R^H$.[25]

In table A.4 I show three distributions that differ in the shape and the average retirement age $\overline{R} = \frac{R^L+R^{mod}+R^H}{3}$. In all three distributions I assume that the earliest retirement age is given by $R^L = 60$. In the first distribution $R^H = 70$ and the modus, median and mean coincide at $R^{mod} = \overline{R} = 65$. In the second distribution the modus is again $R^{mod} = 65$ while the maximum retirement age is $R^H = 67$ implying a mean of $\overline{R} = 64$. Finally, for

[25] I have also analyzed different retirement distributions. The results are robust.

Table A.4: Deficit ratios $d = E/I$

	Distribution 1			Distribution 2			Distribution 3		
Type j	$\hat{\delta} = 0$	$\hat{\delta} = 0.02$	$\hat{\delta} = 0.05$	$\hat{\delta} = 0$	$\hat{\delta} = 0.02$	$\hat{\delta} = 0.05$	$\hat{\delta} = 0$	$\hat{\delta} = 0.02$	$\hat{\delta} = 0.05$
DB	1	1	1.004	1	0.982	0.954	1	1.001	1.003
AR	1	1	1.004	1	0.982	0.954	1	1	1.003
NDC	1	1	1.004	1	0.982	0.954	1	1	1.003

Note: The table shows the deficit ratio for three pension schemes, three assumptions of the discount rate $\hat{\delta} \equiv \delta - g$ and three assumed distributions of the retirement age. These are $R^L = 60$, $R^{mod} = 65$, $R^H = 70$ (distribution 1), $R^L = 60$, $R^{mod} = 65$, $R^H = 67$ (distribution 2) and $R^L = 60$, $R^{mod} = 67$, $R^H = 68$ (distribution 3). The mean retirement age is $\overline{R} = 65$ for distributions 1 and 3 and $\overline{R} = 64$ for distribution 2. Mortality follows a Gompertz distribution with $\alpha = 0.000025$ and $\beta = 0.096$ and the target retirement age is always $R^* = 65$.

the third distribution I assume a non-symmetric case with $\overline{R} = 68$ and $R^{mod} = 67$ which implies an average retirement age of $\overline{R} = 65$.

The first result is that the budget is in balance for all three distributions of retirement ages as long as the discount rate is equal to the growth rate of wages (i.e. $\hat{\delta} = 0$). This is of course an expected result that follows from the analytical findings of section C.4.1. As a second result one can see that the budget is also (approximately) balanced for situations where $\hat{\delta} > 0$ as long as $R^* = \overline{R}$ (which is the case for distributions 1 and 3). For the second distribution, however, for which $R^* = 65 > 64 = \overline{R}$ this is different. In this case the pension system runs a permanent surplus if the discount rate is above the growth rate of wages ($d = 0.98$ for $\hat{\delta} = 0.02$ and $d = 0.95$ for $\hat{\delta} = 0.05$).

These results are completely parallel to the ones of section 3. In particular, the budget is balanced as long as the discount rate is set equal to the growth rate of wages (as has already been shown analytically in section C.4.1). Furthermore, the budget also turns out to be (approximately) balanced for situations where the discount rate differs from this benchmark value as long as $R^* = \overline{R}$.

C.5 Derivation of the pension formulas for the different systems

In this section I derive the formulas for the diferent pension systems that have been stated and used above in section C.2.

C.5.1 Notional defined contribution pension system

I start with the discussion of how the pension level $P(R, a, t)$ is determined in NDC systems. This provides again a useful benchmark case to derive the necessary deductions and supplements for the two other PAYG pension systems (AR and DB).

The contributions in a NDC system are credited to a notional account and they are revalued with a "notional interest rate" $\rho(a, t)$ (that is allowed to change over time and across ages). The total value of this account is called the "notional capital" that accumulates over the working periods of an insured person. When the individual retires at age R the final notional capital is given by:

$$K(R,t) = \int_A^R \tau W(t+x) e^{\int_x^R \rho(s,t+s)\,\mathrm{d}s}\,\mathrm{d}x, \tag{46}$$

where the cumulative factor $e^{\int_x^R \rho(s,t+s)\,\mathrm{d}s}$ indicates how the contribution $\tau W(t+x)$ that is paid into the pension system in period $t+x$ is revalued when calculating the final amount of the notional capital in period $t+R$ (the period of retirement). The notional interest rate is a crucial magnitude in a NDC system as I discuss in a different paper (see Knell 2017*a*). In section C.2 I use two standard definitions that can be found in real-world NDC systems. Both notional interest rates are related to the growth rate of productivity (or of average wages) while one also contains a correction for the fast that the cohort size decreases with the mortality rate $\mu(a)$. The account values of the deceased cohort members are regarded as "inheritance gains" that are distributed among surviving cohort members by granting an extra return $\mu(a)$. In particular:

$$\begin{aligned} \rho(a,t) &= g(t) + \mu(a) \\ &\text{or} \\ \rho(a,t) &= g(t), \end{aligned}$$

which corresponds to equations (32a) and (32b) in section C.2. Using these definitions for the notional interest rate in (46) one can conclude that $K(R,t) = \tau W(t+R)(R-A)$ if one uses the value of $\rho(a,t)$ without inheritance gains (equation (32b)) or $K(R,t) = \tau W(t+R) \int_A^R e^{\int_a^R \mu(s)\,\mathrm{d}s}\,\mathrm{d}a$ if one uses the specification that includes the inheritance gains (equation (32a)). This can also be written as $K(R,t) = \tau W(t+R)(R-A)h(R)$, where $h(R) \equiv \frac{\int_A^R S(a)\,\mathrm{d}a}{(R-A)S(R)}$ as expressed in equation (33). The term $h(R)$ stands for the per capita "inheritance gains premium" to the "normal" notional capital, averaged over the $(R-A)$

contribution periods and distributed among the mass $S(R)$ of surviving members.[26]

The first pension that is received by a member of cohort t who retires at the age R is given by:

$$P_{\mathrm{NDC}}(R,R,t)=\frac{K(R,t)}{e(R)}X_{\mathrm{NDC}}(R,t). \tag{48}$$

The first pension is calculated by taking the final notional capital $K(R,t)$ and turning it into an annual pension payment by using remaining life expectancy $e(R)$ as the annuity conversion factor. In addition, there might also be a deduction factor $X_{\mathrm{NDC}}(R,t)$ that is applied to secure a balanced budget if the formula pension $\frac{K(R,t)}{e(R)}$ is not sufficient. For simplicity I write this demographic deduction factor only as a function of the actual retirement age R and time although—in general—it will also depend on other demographic and policy variables (like R^* and A). The dependence on time is relevant for the case of non-stationary retirement distributions.

From equation (27) it is known that:

$$e(R)=\frac{\int_R^{\omega}S(a)\,\mathrm{d}a}{S(R)}. \tag{49}$$

This can be used together with $K(R,t)=\tau W(t+R)(R-A)h(R)=\tau W(t+R)\frac{\int_A^R S(a)\,\mathrm{d}a}{S(R)}$ to derive the first pension in the case that the notional interest rate includes a correction for the inheritance gains (i.e. $\rho(a,t)=g(t)+\mu(a)$):

$$\begin{aligned} P_{\mathrm{NDC}}(R,R,t) &= \tau W(t+R)\frac{(R-A)h(R)}{e(R)}X_{\mathrm{NDC}}(R,t) \\ &= \tau W(t+R)\frac{\int_A^R S(a)\,\mathrm{d}a}{\int_R^{\omega}S(a)\,\mathrm{d}a}X_{\mathrm{NDC}}(R,t). \end{aligned} \tag{50}$$

This expression is quite intuitive. At the moment of retirement the first pension payment is proportional to the wage level in the period of retirement $t+R$. This is due to the fact that past contributions are indexed to average wage growth. The inclusion of inheritance gains raises the notional capital (which is captured by the expression in the numerator) while the period pension payment depends on remaining life expectancy (which is captured by the expression in the denominator). In addition there might be a correction for early or late retirement $X_{\mathrm{NDC}}(R,t)$.

[26]The relation follows by noting that $e^{\int_a^R \mu(s)\,\mathrm{d}s}$ can be written as: $e^{\int_a^R \mu(s)\,\mathrm{d}s}=e^{\int_0^R \mu(s)\,\mathrm{d}s}e^{\int_0^a -\mu(s)\,\mathrm{d}s}$. Therefore $\int_A^R e^{\int_a^R \mu(s)\,\mathrm{d}s}\,\mathrm{d}a=e^{\int_0^R \mu(s)\,\mathrm{d}s}\int_A^R e^{\int_0^a -\mu(s)\,\mathrm{d}s}\,\mathrm{d}a=(S(R))^{-1}\int_A^R S(a)\,\mathrm{d}a$, where I use (26).

For the situation where inheritance gains are not included and where the notional interest rate is simply given by $\rho(t) = g(t)$ the first pension is:

$$\begin{aligned} P_{NDC'}(R,R,t) &= \tau W(t+R)\frac{R-A}{e(R)}X_{NDC'}(R,t) \\ &= \tau W(t+R)\frac{R-A}{\frac{\int_R^\omega S(a)\,\mathrm{d}a}{S(R)}}X_{NDC'}(R,t). \end{aligned} \tag{51}$$

For the case of rectangular survivorship ($S(a) = 1$ for $a \in [A, \omega]$) one gets that both (50) and (51) lead to the same result that $P_{\text{NDC}}(R,R,t) = \tau W(t+R)\frac{R-A}{\omega-R}X_{\text{NDC}}(R,t)$. This is the same expression that was used in section 2 (see table 1).

Existing pensions are adjusted according to:

$$P_j(R,a,t) = P_j(R,R,t)e^{\int_R^a \vartheta(t+s)\,\mathrm{d}s}, \tag{52}$$

for $a \in [R, \omega^c(t)]$ and where I use the index j since the adjustment in (52) is valid for all three pension systems $j \in \{\text{DB}, \text{AR}, \text{NDC}\}$. The variable $\vartheta(t)$ stands for the adjustment rate in period t and the cumulative adjustment factor $e^{\int_R^a \vartheta(t+s)\,\mathrm{d}s}$ indicates how the first pension $P(R,R,t)$ is adjusted to give the pension payment in period $t+a$. In section C.2 I assume that ongoing pensions are adjusted with the average growth rate of wages as expressed in equation (34) stating that $\vartheta(t) = g(t)$.

In this case one can use (50) and (52) to conclude that with $\rho(a,t) = g(t) + \mu(a)$ the ongoing pension is:

$$P_{\text{NDC}}(R,a,t) = \tau W(t+a)\frac{(R-A)h(R)}{e(R)}X_{\text{NDC}}(R,t),$$

which is equation (35) in section C.2. For $\rho(a,t) = g(t)$ it holds that $P_{NDC'}(R,a,t) = \tau W(t+a)\frac{R-A}{e(R)}\Psi_{NDC'}(R)$. In the following I focus on the first case with a compensation for inheritance gains.

C.5.2 Defined benefit pension system

In a similar vein one can look at the two alternative pension systems discussed in section 2. The defined benefit (DB) system promises a target replacement rate q^* if an individual retires at the target retirement age R^*. The replacement rate is related to average lifetime income, where past incomes are revalued at a rate $\rho(a,t)$ and where there are correction for early/late retirement. In particular, instead of (48) it now holds that:

$$P_{\mathrm{DB}}(R,R,t)=q^{*}\overline{W}^{LT}(R,t)X_{\mathrm{DB}}(R,t), \tag{53}$$

where $\overline{W}^{LT}(R,t)=\frac{\int_A^R W(t+x)e^{\int_x^R \rho(s,t+s)\,\mathrm{d}s}\,\mathrm{d}x}{R-A}$. This expression is closely related to the notional capital (46) for NDC systems. I know of no existing DB system that includes a correction for inheritance gains when indexing past wage levels. Therefore the benchmark DB system is characterized by the indexation $\rho(a,t)=g(t)$. From this is follows that $P_{\mathrm{DB}}(R,R,t)=q^{*}W(t+R)X_{\mathrm{DB}}(R,t)$. As above I assume that existing pension are adjusted with the growth rate of average wages according to (52) and (34) and thus:

$$P_{\mathrm{DB}}(R,a,t)=q^{*}W(t+a)X_{\mathrm{DB}}(R,t),$$

which corresponds to equation (36) in section C.2.

C.5.3 Accrual rate pension system

Finally, one can look at the accrual rate system in the general set-up. The AR system promises a pension that is proportional to the revalued average lifetime income. In particular, for each year of work the system promises a certain percentage κ^{*} (the accrual rate) of this lifetime average that can be claimed at the target retirement age R^{*}. For early retirement there exists a deduction $X_{\mathrm{AR}}(R,t)$. In particular, the first pension is now defined as:

$$P_{\mathrm{AR}}(R,R,t)=\kappa^{*}(R-A)\overline{W}^{LT}(R,t)X_{\mathrm{AR}}(R,t), \tag{54}$$

where $\overline{W}^{LT}(R,t)$ stands for lifetime income (as defined in section C.5.2) for which past incomes are revalued at a rate $\rho(a,t)$. As before and in line with existing AR system I assume that indexation follows the growth rate of average wages, i.e. $\rho(a,t)=g(t)$. From this is follows that $P_{\mathrm{AR}}(R,R,t)=\kappa^{*}W(t+R)(R-A)X_{\mathrm{AR}}(R,t)$. For pension adjustment according to (52) and $\vartheta(t)=g(t)$ one can conclude that:

$$P_{\mathrm{AR}}(R,a,t)=\kappa^{*}W(t+a)(R-A)X_{\mathrm{AR}}(R,t).$$

which corresponds to equation (37) in section C.2.

C.5.4 Demographic adjustment factors

Proposition 2 shows that for a stationary retirement distribution $f(R)$ the NDC system is stable for $X(R,t) = \Psi(R) = 1$. It is now straightforward to discuss the demographic deductions $\Psi_j(R)$ that are necessary to establish balanced budget for pension systems that deviate from the NDC benchmark. This can be seen by looking at equation (38). If the pension payments of the alternative system can be written as $P_j(R,a,t) = \tau W(t+a)\,(\cdots)$ then the correction $\Psi_j(R)$ just has to be set in a way such that it mimics the benchmark NDC-pension given in (41). As a first example one can look at a NDC' system that does not include the compensation for inheritance gains (as is the case for most existing NDC systems with the notable exception of Sweden) and that sets $\rho(a,t) = g(t)$. In this case it has been shown above that the pension is given by $P_{NDC'}(R,a,t) = \tau W(t+a)\frac{(R-A)}{e(R)}\Psi_{NDC'}(R)$. It is immediately apparent that an adjustment with $\Psi_{NDC'}(R) = h(R)$ leads to a balanced budget. Otherwise, the pension system would run a surplus since $h(R) > 1$, i.e. the system would keep the inheritance gains for itself instead of distributing them among the surviving population.

For the defined benefit system I invoke as in section 2 the "balanced target condition", i.e. I assume that the target replacement rate q^* is associated with a situation that there will not be any deductions ($\Psi_{\mathrm{DB}}(R^*) = 1$) if everybody retires at the target retirement age R^*. For the NDC system one knows from equation (41) that a balanced budget with $R = R^*, \forall i$ requires that everybody gets a pension equal to $P_{\mathrm{NDC}}(R^*,a,t) = \tau W(t+a)\frac{(R^*-A)h(R^*)}{e(R^*)}$. This should be equal to the DB pension with $R = R^*, \forall i$, i.e. to $P_{\mathrm{DB}}(R^*,a,t) = q^* W(t+a)$. From these two expressions it follows that $q^* = \tau\frac{(R^*-A)h(R^*)}{e(R^*)}$. Inserting this into equation (36) for $P_{\mathrm{DB}}(R,a,t)$ leads to the DB pension after evoking the stability condition:

$$P_{\mathrm{DB}}(R,a,t) = \tau W(t+a)\frac{(R^*-A)h(R^*)}{e(R^*)}\Psi_{\mathrm{DB}}(R). \tag{55}$$

This expression can now be compared to the pension of the benchmark NDC system (41) (that leads to a balanced budget) to conclude that $\Psi_{\mathrm{DB}}(R) = \frac{R-A}{R^*-A}\frac{e(R^*)}{e(R)}\frac{h(R)}{h(R^*)}$ as stated in equation (42).

One can use similar steps for the AR pension system. In particular, I assume that the target accrual rate κ^* is chosen in such a way that the system is balanced if everybody retires at the target retirement age $R = R^*, \forall i$. Inserting the implied target accrual rate $\kappa^* = \tau\frac{h(R^*)}{e(R^*)}$ into equation (37) for $P_{\mathrm{AR}}(R,a,t)$ leads to the AR pension after evoking the stability condition:

$$P_{\mathrm{AR}}(R,a,t)=\tau W(t+a)\frac{(R-A)h(R^*)}{e(R^*)}\Psi_{\mathrm{AR}}(R). \tag{56}$$

The deduction rate that is necessary for a balanced AR system can be calculated by setting (56) equal to the NDC pension (41) and solving for $\Psi_{\mathrm{AR}}(R)$. This leads to $\Psi_{\mathrm{AR}}(R) = \frac{e(R^*)}{e(R)}\frac{h(R)}{h(R^*)}$ as stated in equation (43). These and other important formulas are collected in table A.1 of section C.2.

D Extensions (section 5)

In the main part of the paper I have assumed that all individuals have the same entry age A, the same wage level W and the same life expectancy ω. In this part of the paper I broaden up this framework and now assume that the society is split in $j = 1, 2, \dots J$ groups (with respective weight π_j) where each group is characterized by specific values for A_j, W_j, ω_j and possibly also R_j^* (i.e. the system might prescribe group-specific target retirement ages).[27] A possible correlation between income, entry age, life expectancy etc. is captured by the pattern of relative frequencies π_j of the different groups with $\sum_{j=1}^{J}\pi_j = 1$. Although members of each group are identical along all the above mentioned dimensions they might differ in their retirement age. In particular, I assume that within each group j there are $k = 1, 2, \dots K$ subgroups of individuals with a retirement age R_j^k and a relative frequency ϕ_j^k with $\sum_{k=1}^{K}\phi_j^k = 1$. The within-group differences in the retirement age are important in order to calculate the appropriate deductions for early retirement at the age of R_j^k instead of R_j^*.

Throughout the following analyses I assume that the formula of the NDC system is based on average life expectancy $\overline{\omega} = \sum_{j=1}^{J}\pi_j\omega_j = 1$ rather than on group-specific life expectancy ω_j. This might be due to reasons of information availability or political choice. It will become apparent that this fact has major budgetary and distributional implications while heterogeneity in the other variables does not affect the main conclusions.

[27] In fact, one could also introduce heterogeneous wage profiles. Assume that group j of the cohort born in period t earned a wage $W_j(a,t) = \xi_j(a,t)W(t+a)$ in each of its working period between age A_j and age R_j where $W(t)$ now is the aggregate wage in period t. For the NDC system the notional capital would then be given by $K_j(R_j,t) = \tau W(t+R)(R_j - A_j)\overline{EP}_j$, where $\overline{EP}_j \equiv \left(\int_{A_j}^{R_j}\xi_j(a,t+a)\,\mathrm{d}a\right)/(R_j - A_j)$ stands for the average lifetime "earnings points" (to use an expression from the German pension system). If group j had always earned average wages then $\xi_j(a,t+a) = 1, \forall a$ and $\overline{EP}_j = 1$.

D.1 Homogeneous life expectancy (sections 5.2, 5.3)

I start with the case of homogeneous life expectancy, i.e. $\omega_j = \overline{\omega}$. I now allow, however, as described in section 5.3 for the case of a "biased NDC system", i.e. $\widehat{P}_j^k = \tau W_j \eta_j \frac{R_j^k - A_j}{\overline{\omega} - R_j^k}$.

For the benchmark case with $R_j^k = R_j^*$ for all members k, revenues and expenditures are given by $I_j^* = \tau W_j (R_j^* - A_j)$ and $E_j^* = P_j^* (\overline{\omega} - R_j^*) = \tau W_j \eta_j (R_j^* - A_j)$ (where I normalize the cohort size to $N_j = 1$). The deficit therefore comes out as $D_j^* = E_j^* - I_j^* = \tau W_j (\eta_j - 1)(R_j^* - A_j)$ and the deficit ratio as $d_j^* = E_j^* / I_j^* = \eta_j - 1$.

Members of group j differ, however, in their retirement ages R_j^k. One can use the same steps as in section 2.2 to calculate the appropriate level of deductions X_j^k that leaves the present value of payment streams unchanged for early or late retirement. This leads to:

$$X_j^k = 1 + \frac{R_j^* - R_j^k}{R_j^k - A_j} \frac{\eta_j - 1}{\eta_j} - \frac{\delta}{2}(R^* - R_j^k) \frac{\overline{\omega} - R_j^* + \eta_j (R_j^* - A_j)}{\eta_j (R_j^k - A_j)}. \qquad (57)$$

For $\eta_j = 1$ this is the same expression as (2) for the NDC system. The influence of η_j on the budget-neutral deduction is not huge but still non-negligible. E.g., for $\delta = 0$, $A_j = 20$, $\overline{\omega} = 80$, $R_j^* = 65$ and $\eta_j = 1.25$ one gets that $X_j^k = 1.0046$ (for $R_j^k = 64$). The early retiree receives an extra *supplement* of about 0.5%. The reason for this is that the normal NDC adjustment for early retirement lowers the annual pension payment and the multiplicative reward η_j is thus applied to a smaller "base" which would—in the absence of X_j^k—reduce the total pension payments for early retirees in group j.

Total revenues for the group j are given by $I_j = \sum_{k=1}^K \phi_j^k \tau W_j (R_j^k - A_j) = \tau W_j (\overline{R}_j - A_j)$, where $\overline{R}_j \equiv \sum_{k=1}^K \phi_j^k R_j^k$ stands for the average retirement age among the members of group j. Total expenditures, on the other hand, are given by $E_j = \sum_{k=1}^K (\overline{\omega} - R_j^k) P_j^k$, where $P_j^k = \widehat{P}_j^k X_j^k$ is the pension payment associated with retirement age R_j^k (including the additional deduction X_j^k). Using (57) one can conclude that (for $\delta = 0$) $(\overline{\omega} - R_j^k) P_j^k = \tau W_j (\eta_j (R_j^k - A_j) + (R_j^* - R_j^k)(\eta_j - 1))$ and thus $E_j = \tau W_j (\eta_j (\overline{R}_j - A_j) + (R_j^* - \overline{R}_j)(\eta_j - 1))$. It follows that $D_j = E_j - I_j = \tau W_j (\eta_j - 1)(R_j^* - A_j)$ which is the same as in the benchmark with $R_j^k = R_j^*, \forall k$. As far as the deficit ratio is concerned one has to note that total revenues are given by $I_j = \tau W_j (\overline{R}_j - A_j)$. For $R_j^* = \overline{R}_j$ it is the case that $d_j = (\eta_j - 1)$ (which is the same as in the benchmark). For a homogeneous target age $R_j^* = R^*$, however, this is only possible if the mean retirement age is the same across all groups j which cannot be taken for granted.

The size of the overall deficit $\overline{D} = \sum_{j=1}^J \pi_j D_j$ depends on the correlation between variables A_j, W_j and R_j^* since $\overline{D} = \sum_{j=1}^J \pi_j \tau W_j (\eta_j - 1)(R_j^* - A_j)$. In the absence of a ‘bias” (i.e for η_j=1) one can observe, however, that these possible correlations do not matter since each subgroup will have a balanced budget with $D_j = 0$ and thus also the overall budget $\overline{D}$ will be balanced.

The “bonus” (or “social security wealth”) of the “biased NDC system” to different individuals k in group j is given by:

$$B_j^k = (\overline{\omega} - R_j^k)P_j^k - \tau W_j(R_j^k - A_j) = \tau W_j(\eta_j - 1)(R_j^* - A_j).$$

This is the same for different members of group j (i.e. it does not vary by the individual retirement age R_j^k).

The use of the “normal” deduction factor $X_j^k = 1$ instead of (57) leads to $(\overline{\omega} - R_j^k)P_j^k = \tau W_j \eta_j (R_j^k - A_j)$ and thus $E_j = \tau W_j \eta_j (\overline{R}_j - A_j)$ and $D_j = \tau W_j(\eta_j - 1)(\overline{R}_j - A_j)$. This is the same deficit as in the benchmark. Now, however, the “bonus” that the system pays to different individuals is not identical for all members k but rather $B_j^k = \tau W_j(\eta_j - 1)(R_j^k - A_j)$. The difference is not huge (about 2.3% for $R_j^k = 64$ and 12.5% for $R_j^k = 60$), but certainly not completely irrelevant.

D.2 Heterogeneous life expectancy (section 5.4)

I turn now to the general case with group-specific life expectancy ω_j (but where—for the simplicity—I abstract from an additional possible bias η_j). I assume that the group-specific life expectancy is known while the particular longevity of a member k of group j is still uncertain. All following derivation and arguments thus refer to the representative member of the group and I disregard uncertainty. The pension for member k of group j is given by $\widehat{P}_j^k = \tau W_j \frac{R_j^k - A_j}{\overline{\omega} - R_j^k}$, where it is again assumed that the NDC pension is based on average life expectancy $\overline{\omega}$ instead of the group-specific value ω_j (despite the assumption that the group-specific magnitudes can or could be observed).

For the benchmark case with $R_j^k = R_j^*$ for every member k, revenues and expenditures are given by $I_j^* = \tau W_j(R_j^* - A_j)$ and $E_j^* = P_j^*(\omega_j - R_j^*) = \tau W_j(\omega_j - R_j^*)\frac{R_j^* - A_j}{\overline{\omega} - R_j^*}$. The deficit comes out as:

$$D_j^* = E_j^* - I_j^* = \tau W_j(R_j^* - A_j)\frac{\omega_j - \overline{\omega}}{\overline{\omega} - R_j^*} \tag{58}$$

and the deficit ratio as $d_j^* = \frac{\omega_j - \overline{\omega}}{\overline{\omega} - R_j^*}$. This shows that the use of average life expectancy implies that some groups might be favored by the NDC system. In particular, groups with a higher-than-average life expectancy ($\omega_j > \overline{\omega}$) will get out more from the pension system than they have paid (i.e. $d_j^* > 0$). This is thus a different situation than caused by the use of a uniform "bonus" $\eta_j = \overline{\eta}$ that is the same for every group j independent of specific group characteristics like life expectancy ω_j.[28]

Using the same steps as in section 2.2 one can again calculate the appropriate level of deductions X_j^k that leave the net present value unchanged for early or late retirement:

$$\begin{aligned} X_j^k &= \frac{(\overline{\omega} - R_j^k)\left(R_j^k(\overline{\omega} - R_j^*) - A_j(\omega_j - R_j^*) + R_j^*(\omega_j - \overline{\omega})\right)}{(R_j^k - A_j)(\overline{\omega} - R_j^*)(\omega_j - R_j^k)} \\ &\quad -\frac{\delta}{2}(R^* - R)\frac{(\overline{\omega} - A_j)(\overline{\omega} - R_j^k)(\omega_j - R^*)}{(R_j^k - A_j)(\overline{\omega} - R^*)(\omega_j - R_j^k)}. \end{aligned} \tag{59}$$

For $\omega_j = \overline{\omega}$ equation (59) reduces to $X_j^k = 1 - \frac{\delta}{2}\left(R_j^* - R_j^k\right)\frac{\overline{\omega} - A_j}{R_j^k - A_j}$, which corresponds to expression Δ in section 2.

For $\delta = 0$ the "bonus" (or "malus") of the system when using X_j^k can be calculated as:

$$\begin{aligned} B_j^k &= P_j^k(\omega_j - R_j^k) - \tau W_j(R_j^k - A_j) \\ &= \tau W_j(R_j^k - A_j)\frac{\omega_j - R_j^k}{\overline{\omega} - R_j^k}\left(\frac{(\overline{\omega} - R_j^k)\left(R_j^k(\overline{\omega} - R_j^*) - A_j(\omega_j - R_j^*) + R_j^*(\omega_j - \overline{\omega})\right)}{(R_j^k - A_j)(\overline{\omega} - R_j^*)(\omega_j - R_j^k)}\right) \\ &\quad -\tau W_j(R_j^k - A_j) \\ &= \tau W_j(R_j^* - A_j)\frac{\omega_j - \overline{\omega}}{\overline{\omega} - R_j^*}. \end{aligned} \tag{60}$$

The bonus/supplement of the system is thus the same for every member k of group j (or—put differently—the size of the bonus does not depend on the individual retirement age R_j^k but only on the characteristics of the group j).

The deficit of group j is given by $D_j = \sum_{k=1}^{K} \phi_j^k D_j^k = \sum_{k=1}^{K} \phi_j^k B_j^k$. It comes out as:

$$D_j = \tau W_j(R_j^* - A_j)\frac{\omega_j - \overline{\omega}}{\overline{\omega} - R_j^*}. \tag{61}$$

This is the same as for the benchmark case with $R_j^k = R_j^*$ (see equation (58)). The deficit *ratio* will only be the same for $R_j^* = \overline{R}_j$ which (as discussed above) cannot be taken for

[28] In fact, one could also formulate this case a model where the NDC uses individual life expectancy ω_j but involves group-specific bonuses η_j. In particular, one could write $\widehat{P}_j^k = \tau W_j \eta_j \frac{R_j^k - A_j}{\omega_j - R_j^k}$ where $\eta_j = \frac{\omega_j - R_j^k}{\overline{\omega}_j - R_j^k}$ or $\eta_j - 1 = \frac{\omega_j - \overline{\omega}}{\overline{\omega}_j - R_j^k}$.

granted. Whether the overall deficit $\overline{D} = \sum_{j=1}^{J} \pi_j D_j$ will be balanced depends on the correlation between the main variables A_j, W_j, ω_j (and R_j^*).

For homogeneous life expectancy ($\omega_j = \overline{\omega}$) equation (61) implies $D_j = 0$ for every group and thus also the overall deficit is zero. As was the case for $\eta_j = 1$, also for homogeneous life expectancy the other heterogeneous elements and their possible correlation do not have an effect on the budget and on the interpersonal distribution (since also $B_j^k = 0, \forall k$). For heterogeneous life expectancy, however, the deficit is no longer zero for every group. Equation (61) rather implies that $D_j \gtreqless 0$ for $\omega_j \gtreqless \overline{\omega}$. Furthermore, it is not the case that the surpluses and deficits for the various groups just cancel. In particular, $\overline{D} = \sum_{j=1}^{J} \pi_j \tau W_j (R_j^* - A_j) \frac{\omega_j - \overline{\omega}}{\overline{\omega} - R_j^*}$. If $R_j^* = R^*$ and the variables are uncorrelated then one can conclude that $\overline{D} = 0$ (a particular instance of this is naturally the case where $A_j = A$ and $W_j = W$). In general, however, this is not the case (even not for $R_j^* = R^*$). If, e.g., the long-lived groups are also the ones with high wages (W_j and ω_j are positively correlated) then the surpluses of the short-lived will not counterbalance the deficits of the long-lived.

It is interesting to have a closer look at the deduction factor X_j^k in (59) and how it is affected by the individual retirement age and individual life expectancy. One can calculate that:

$$\frac{\partial X_j^k}{\partial R_j^k} = \frac{(\overline{\omega} - A_j)(\overline{\omega} - \omega_j)\left((R_j^* - A_j)(\omega_j - R_j^*) + (R_j^* - R_j^k)^2\right)}{(R_j^k - A_j)^2(\overline{\omega} - R_j^*)(\omega_j - R_j^k)^2}, \tag{62}$$

$$\frac{\partial X_j^k}{\partial \omega_j} = \frac{(\overline{\omega} - A_j)(\overline{\omega} - R_j^k)(R_j^* - R_j^k)}{(R_j^k - A_j)(\overline{\omega} - R_j^*)(\omega_j - R_j^k)^2}. \tag{63}$$

One can thus conclude that $\frac{\partial X_j^k}{\partial R_j^k} \gtreqless 0$ for $\overline{\omega} \gtreqless \omega_j$ while $\frac{\partial X_j^k}{\partial \omega_j} \gtreqless 0$ for $R_j^* \gtreqless R_j^k$. An example is helpful to illustrate this. Assume $R_j^* = 65$ and $\overline{\omega} = 80$. Then for $\omega_j = 84$ one gets $X_j^k = 0.94$ (for $R_j^k = 68$) and $X_j^k = 1.05$ (for $R_j^k = 62$) while for $\omega_j = 76$ one gets $X_j^k = 1.13$ (for $R_j^k = 68$) and $X_j^k = 0.92$ (for $R_j^k = 62$). The intuition is similar as was the case for the biased NDC system. The long-lived groups $\omega_j > \overline{\omega}$ are treated favorably by the system. If someone of this group retires early then the "bonus" would be smaller than for the benchmark individual with $R_j^k = R_j^*$. This is compensated with a $X_j^k > 1$. For $\omega_j < \overline{\omega}$ the argument is reversed. By its design (i.e. the use of average life expectancy for annuitization) the system has decided to give different bonuses or maluses according to life expectancy. Irrespective of whether this redistribution is regarded as reasonable, the retirement age should not have an affect on the amount of these transfers. This is achieved by the use of X_j^k.

In reality, NDC systems typically use average life expectancy for calculating the annuity but I know of no system that in fact has implemented a corrective deduction factor like X_j^k in equation (59). It is thus interesting to look at the consequences if a NDC system just uses the basic formula $P_j^k = \tau W_j \frac{R_j^k - A_j}{\overline{\omega} - R_j^k}$ without an additional X_j^k. Straightforward calculations show that the bonus is in this case $B_j^k = \tau W_j (R_j^k - A_j) \frac{\omega_j - \overline{\omega}}{\overline{\omega} - R_j^k}$. This has the following implications. First, the overall deficit D_j of group j is no longer independent of the retirement behavior. This only holds (approximately) true if $R_j^* = \overline{R}_j$. This, however, cannot be taken for granted. In particular, it implies that the retirement age is not correlated with lifetime wages, entry age and life expectancy which is implausible. Second, different to (60) now the bonus is not the same for every member k of group j. Rather it holds that $\frac{\partial B_j^k}{\partial \omega_j} > 0$ (long-lived get higher bonuses) and $\frac{\partial B_j^k}{\partial R_j^k} \gtreqless 0$ for $\omega_j \gtreqless \overline{\omega}$. It is not clear whether this is a desired consequence. An early retiree among the long-lived will now get a smaller bonus than before (which might be an appreciated consequence), but a long-lived late retiree will be rewarded with an even larger bonus (which looks less favorable). On the other hand, the early short-lived retiree has a smaller disadvantage than with the use of X_j^k (which might be regarded as positive) but the late retirees among the short-lived get a larger punishment.

D.3 Pension adjustment with inflation (section 5.3)

I have assumed in the model of appendix C that ongoing pensions are adjusted with the rate of growth ($\vartheta(t) = g(t)$) as shown in equation (34). In reality there exist different adjustment regimes and while some countries use growth rate of average wages for adjustment others refer to the growth rate of the wage sum, the rate of inflation or mixtures of these concepts. It is thus interesting to study the implications of different adjustment regimes. I focus on the case where pensions are held constant in real terms, i.e. where they are are only indexed to the rate of inflation and thus $\vartheta(t) = 0$. For the NDC system this means that:

$$\widehat{P}_{\text{NDC}}(R, a, t) = \widehat{P}_{\text{NDC}}(R, R, t) \;=\; \eta \tau W(t+R) \frac{(R-A)h(R)}{e(R)} = \eta \tau W(t+R) \frac{\int_A^R S(a)\,\mathrm{d}a}{\int_R^\omega S(a)\,\mathrm{d}a},$$

where I allow for an additional multiplier η to the "pure" NDC pension as discussed above in section D.1. For the benchmark case with $R = R^*$ one gets that $I^* = \tau W(t) \int_A^{R^*} S(a)\,\mathrm{d}a$

and $E^* = \int_{R^*}^{\omega} P(R^*, a, t-a) S(a)\,\mathrm{d}a$. For constant wage growth $g(t) = g$ this can be written as $E^* = \eta\tau W(t) \frac{\int_A^{R^*} S(a)\,\mathrm{d}a}{\int_{R^*}^{\omega} S(a)\,\mathrm{d}a} \int_{R^*}^{\omega} e^{-g(a-R^*)} S(a)\,\mathrm{d}a$. For a balanced budget in the benchmark situation one can thus require that:

$$\eta = \eta^* \equiv \frac{\int_{R^*}^{\omega} S(a)\,\mathrm{d}a}{\int_{R^*}^{\omega} e^{-g(a-R^*)} S(a)\,\mathrm{d}a}.$$

For $R^* = 65$ and a Gompertz mortality distribution with $\alpha = 0.000025$ and $\beta = 0.096$ (implying a life expectancy at birth of around 80) one gets that $\eta^* = 1.24$ (for $g = 0.02$), $\eta^* = 1.12$ (for $g = 0.01$) and $\eta^* = 1.06$ (for $g = 0.005$). The system is saving money due to the lower pension adjustment and thus could increase the first pensions considerably (between 5% and 25%). If this mark-up is absent (i.e. $\eta = 1$) then the NDC system would run considerable surpluses (also between 5% and 25% each period).

The deductions are only slightly affected by this different adjustment regime. It can be shown that for $\eta = 1$ the budget-neutral deduction rate is between $x = 0.002$ (for $g = 0.02$) and $x = 0.0013$ (for $g = 0.01$), while for $\eta = \eta^*$ they are somewhat higher, between $x = 0.007$ (for $g = 0.02$) and $x = 0.004$ (for $g = 0.01$). They are again positive for the same reasons as discussed in section D.1.

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