OESTERREICHISCHE NATIONALBANK
EUROSYSTEM

WORKING PAPER 150

How to find plausible, severe, and useful stress scenarios

Thomas Breuer, Martin Jandačka, Klaus Rheinberger and Martin Summer

Statement of Purpose

The Working Paper series of the Oesterreichische Nationalbank is designed to disseminate and to provide a platform for discussion of either work of the staff of the OeNB economists or outside contributors on topics which are of special interest to the OeNB. To ensure the high quality of their content, the contributions are subjected to an international refereeing process. The opinions are strictly those of the authors and do in no way commit the OeNB.

Imprint: Responsibility according to Austrian media law: Günther Thonabauer, Secretariat of the Board of Executive Directors, Oesterreichische Nationalbank
Published and printed by Oesterreichische Nationalbank, Wien.
The Working Papers are also available on our website (http://www.oenb.at) and they are indexed in RePEc (http://repec.org/).

Editorial

The authors give a precise operational definition to three requirements the Basel Committee on Banking Supervision specifies for stress tests: Plausibility and severity of stress scenarios as well as suggestiveness of risk reducing actions. The basic idea of the authors' approach is to define a suitable region of plausibility in terms of the risk factor distribution and search systematically for the worst portfolio loss over this region. One key innovation compared to the existing literature is the solution of two open problems. The authors suggest a measure of plausibility that is not prone to the problem of dimensional dependence of maximum loss and they derive a way to consistently deal with situations where some but not all risk factors are stressed. Among the various approaches used for partial scenarios, plausibility is maximised by setting the non stressed risk factors to their conditional expected value given the value of the stressed risk factors.

February 5, 2009

How to find plausible, severe, and useful stress scenarios*

THOMAS BREUER MARTIN JANDAČKA
KLAUS RHEINBERGER MARTIN SUMMER

Abstract

We give a precise operational definition to three requirements the Basel Committee on Banking Supervision specifies for stress tests: Plausibility and severity of stress scenarios as well as suggestiveness of risk reducing actions. The basic idea of our approach is to define a suitable region of plausibility in terms of the risk factor distribution and search systematically for the worst portfolio loss over this region. One key innovation compared to the existing literature is the solution of two open problems. We suggest a measure of plausibility that is not prone to the problem of dimensional dependence of maximum loss and we derive a way to consistently deal with situations where some but not all risk factors are stressed. Among the various approaches used for partial scenarios, plausibility is maximised by setting the non stressed risk factors to their conditional expected value given the value of the stressed risk factors.

Keywords: Stress testing, maximum loss, risk management, banking regulation.
JEL-Classification Numbers: G28, G32, G20, C15.

*Thomas Breuer, Martin Jandačka and Klaus Rheinberger, Research Centre PPE, Fachhochschule Vorarlberg, Hochschulstr. 1, A-6850 Dornbirn, Austria. Martin Summer, Oesterreichische Nationalbank, Otto Wagner Platz 3, A-1090 Vienna, Austria. Corresponding author: thomas.breuer(at)fhv.at. MJ and KR are supported by the Internationale Bodenseehochschule. We are grateful to Helmut Elsinger, Gerald Krenn, Bjørne Dyre Syversten and Nikola Tarashev for comments.

1 Introduction

The current regulatory framework of the Basel Committee on Banking Supervision [2005] requires banks to perform stress tests which meet three requirements: *plausibility* of stress scenarios, *severity* of stress scenarios and *suggestiveness of risk reducing action*.[1] How do we find stress scenarios which are the same time plausible, severe, and suggestive for the design of risk reducing action? Our paper gives a systematic answer to this question. We suggest a method that can be implemented for a wide class of stress testing problems usually encountered in practice. We illustrate the method and the issues in the context of an example: stress tests for a portfolio of adjustable rate loans in home and foreign currency.

The quality of a stress test crucially depends on the definition of stress scenarios. Defining stress scenarios is a thought experiment. It is a counterfactual exercise where a risk manager tries to imagine what adverse or even catastrophic events might strike his portfolio. Such a thought experiment is prone to two major pitfalls: consideration of implausible scenarios and neglection of plausible scenarios. Thinking about scenarios requires to imagine situations that have not yet occurred but might occur in the future. Bias towards historical experience can lead to the risk of ignoring plausible but harmful scenarios which did not yet happen in history. This creates a dangerous blind spot. If the imagination of a stress tester puts excessive weight on very unplausible scenarios management faces an embarrassing decision: should one react to alarming results of highly implausible stress scenarios? Our method allows a precise trade off between plausibility and severity. In this way we can ensure that in a model of portfolio risk, no harmful but plausible scenarios are missed. Furthermore, our stress test method suggests ways to reduce risk if desired.

We analyse the problem of finding extreme but plausible scenarios in a classical quantitative risk management framework, assuming some statistical model. A portfolio of financial instruments, say a portfolio of loans, is given. The value of each loan at some given horizon in the future is described by the realization of certain risk factors. In the case of a loan portfolio, for example, these risk factors will comprise the macroeconomic environment (because of its impact on the payment ability and thus on the solvency of borrowers), market factors like interest rates (or exchange rates in the case of foreign currency loans) but also idiosyncratic factors that influence a borrower's solvency. The uncertainty about the realization of risk factors is described by a risk factor distribution that is estimated from historical data. Plausibility is captured by specifying how far we go into the tails of the distribution in our search for stress scenarios. The severity of scenarios is maximized by systematically searching for the worst case, the maximum portfolio loss, in a risk factor region of given plausibility.

This general idea of looking at extreme scenarios has been formulated in the literature before. In the context of credit risk stress testing, it is informally discussed by Čihák [2004, 2007]). For market risk stress tests, the idea is discussed

[1] The respective references in Basel Committee on Banking Supervision [2005] are: "Quantitative criteria should identify plausible stress scenarios to which banks could be exposed." (par. 718 (LXXIX))," (...) a bank should also develop its own stress tests which it identifies as most adverse based on the characteristics of its portfolio" (par. 718 (LXXXIII)) and "Qualitative criteria should emphasize that two major goals of stress testing are to evaluate the capacity of the bank's capital to absorb potential large losses and to identify steps the bank can take to reduce its risk and conserve capital." (par. 718(LXXIX))

more formally in Studer [1999, 1997] and in Breuer and Krenn [1999]. This literature leaves however two open issues that seem technical at first sight but are of great practical relevance: The problem of partial scenarios and the problem of dimensional dependence of maximum loss. In another respect this paper is novel. It performs stress tests integrating market and credit risk, instead of just one of the two. It will turn out that the interaction effects are considerable.

The partial scenario problem comes from the situation that a portfolio may depend on many risk factors but the modellers are interested in stressing not all but only a few factors at a time. For example in a loan portfolio we are often interested in stress scenarios for particular variables: A certain move in the exchange rate, or a particular drop in GDP. How do we deal with the other risk factors consistently? Do we leave them at their last observed value, at some average value, should we condition on the stressed macro factor and if so how? We show that the way to deal with the partial scenario problem that maximizes plausibility is to set the non stressed systematic risk factors to their conditional expectation for the given value of the stressed factors. We show furthermore that this has the same plausibility than the computationally more intensive full loss simulation from the conditional stress distribution as in Bonti et al. [2005].

If we look for maximum loss in a risk factor region of given plausibility we want the maximum loss to be independent from the inclusion of irrelevant risk factors or risk factors that are highly correlated with factors already included in the analysis. The plausibility measures that were used in the previous literature (see Studer [1999, 1997]) suggested to define plausibility regions as regions with a given probability mass. This definition of plausibility has an undesirable property, known as the problem of dimensional dependence of maximum loss. The reason is that the probability mass of an n-dimensional ellipsoid of given radius depends on the number of dimensions, in the same way as the volume of an n-sphere depends on the nunber of dimensions. In a higher dimensional space the radius needs to be larger in order for the ellipsoid to contain the same probability mass. If one defined plausibility of a scenario in terms of the probability mass of the ellipsoid of smaller moves, then maximum loss would depend on the number of risk factors. This number is to some degree arbitrary because models which include or exclude risk factors which are completely or almost irrelevant to the portfolio value lead to the same profit loss distribution and cannot or only hardly be distinguished empirically. In a higher dimensional represenation the ellipsoid will have a larger radius. It therefore allows for larger risk factor moves and thus for more harmful stress scenarios with higher loss.

To get an intuitive understanding of the problem consider an example from Breuer [2008]. We have a bond portfolio with risk factors consisting of two yield curves in 10 foreign currencies. One risk manager chooses to model the yield curve with seven maturity buckets and another risk manager uses 15 buckets. In this case the first risk manager uses 150 risk factors in his analysis and the second manager uses 310. As plausibility region both of them choose an ellipsoid of mass 95%. Breuer [2008] shows that the second risk manager will calculate a maximum loss that is 1.4 times higher than the maximum loss calculated by first risk manager. This is problematic because both of them look at the same portfolio and use the same plausibility level. Our measure of plausibility does not have this problem.

The paper is organized as follows: In Section 2 we define a quantitative measure of plausibility and explain why it is not subject to the dimensional dependence problem. We discuss how to deal with the problem of partial scenarios and explain the technique of worst case analysis. We also discuss how measures for risk reducing actions can be deduced from the stress test. In Section 3 we analyse an example of a portfolio of foreign currency loans that illustrates the practical applicability as well as the potential improvement compared to a standard stress testing procedure. The final section (4) concludes.

2 Finding scenarios that are plausible, severe, and suggestive of counter-action

We consider the problem of stress testing a loan portfolio. The value of each position in the portfolio depends on n systematic risk factors $\boldsymbol{r} = (r_1, \ldots, r_n)$ and on m idiosyncratic risk factors $\epsilon_1, \ldots, \epsilon_m$. Typically, the systematic risk factors are market and/or macroeconomic risk factors, and the idiosyncratic risk factors refer to the individual counterparties. The number of idiosyncratic risk factors is much larger than the systematic risk factors. In our approach we have to restrict the distribution of the systematic risk factors $\boldsymbol{r}$ to a class called the elliptical distributions. For the definition and some basic facts about elliptical distributions we refer to the standard work of Fang et al. [1987]. For our purpose it is enough to note that the standard distributions used in classical risk management problems are in fact from this class. We denote the covariance matrix and expectations of the distribution of $\boldsymbol{r}$ by Cov and $\boldsymbol{\mu}$. The distribution of the idiosyncratic risk factors may be arbitrary as long as the expectation of the portfolio value function, conditional on some or all systematic risk factors, exists.

2.1 Plausible scenarios

In a stress test of a loan portfolio we imagine extreme realizations of one or more of the systematic risk factors. How would we quantify the plausibility of this thought experiment?

An intuitive approach could be to compare the extreme realisation of a risk factor to its average. Intuitively the further we are away from this average value, the less plausible the stress scenario becomes. The distance should be measured in standard deviations. For multi-variate moves the plausibility should depend additionally on the correlations. A multi-variate move which is in agreement with the correlations is more plausible than a move against the correlations.

A statistical concept that formalises these ideas is the so called Mahalanobis distance given by

$$\mathrm{Maha}(\boldsymbol{r}) := \sqrt{(\boldsymbol{r} - \boldsymbol{\mu})^T \cdot \mathrm{Cov}^{-1} \cdot (\boldsymbol{r} - \boldsymbol{\mu})},$$

The Mahalanobis distance is simply the distance of the test point from the center of mass divided by the width of the ellipsoid in the direction of the test point. Intuitively, $\mathrm{Maha}(\boldsymbol{r})$ can be interpreted as the number of standard deviations of the multivariate move from $\boldsymbol{\mu}$ to $\boldsymbol{r}$. Maha takes into account the correlation structure and the standard deviations of the risk factors.

Although Maha is an intuitive measure of plausibility, it has two drawbacks. First, it reflects only the first two moments of the risk factor distribution because it only depends on $\boldsymbol{\mu}$ and Cov. For risk factor distribution agreeing on the first two moments but not on the higher moments, a given move will have the same Maha although that move might be more plausible for the distribution with fatter tails. Second, Maha depends on the choice of coordinates Breuer [2008]. There are transformations mapping the normal into the normal distribution, but not mapping the ellipsoid into the ellipsoid. Coordinate dependence confronts us with the dilemma of choosing one of two undesirable alternatives. Either we single out one specific coordinate system and take as admissibility domain in other coordinate systems the transform of the ellipsoid, even if the transform is not an ellipsoid. But it is unnatural to take a non-elliptical admissibilty domain in transformed coordinates, in which risk factors are distributed normally. Alternatively we can take as admissibility domain in the transformed coordinates the ellipsoids. But this will result in a different worst case scenario and in a different MaxLoss value than in the original coordinates. Both drawbacks can be avoided with a more general notion of plausibility, which uses generalised scenarios.[2]

In contrast to the previous literature we define plausibility directly in terms of Maha($\boldsymbol{r}$): A high value of Maha implies a low plausibility of the scenario $\boldsymbol{r}$. Earlier work defined plausibility in terms of the *probability mass* of the ellipsoid of all scenarios of equal or lower Maha, see Studer [1999, 1997] or Breuer and Krenn [1999]. That approach creates the problem of dimensional dependence of maximum loss. In our approach this problem does not occur because we specifiy the size of the ellipsoid in terms of its Maha-radius, not it terms of its probability mass. Breuer [2008] proves that this notion of plausibility overcomes the problem of dimensional dependence of maximum loss.

2.2 Partial scenarios

Typically portfolios are modelled with hundreds or thousands of risk factors. Stress scenarios involving the full plethora of risk factors are hardly tractable numerically and overwhelmingly complex to interpret. Therefore practioners use only scenarios with a handful of risk factors. These we call partial scenarios, because not all risk factors of the model are stressed. How should the other risk factors be treated?

Kupiec [1998] discussed four different ways to deal with the risk factors not fixed by some partial scenario:

(A) The other systematic risk factors remain at their *last observed* value.

(B) The other macro risk factors take their *unconditional expectation* value.

(C) The other systematic risk factors take their *conditional expected value* given the values of the fixed risk factors. Denote by $\boldsymbol{r}_C$ the resulting vector of values of the systematic risk factors.

[2] Generalised scenarios are probability distributions rather than points of the sample space. The plausibility of a generalised scenario can be measured by the relative entropy of this distribution with respect to the prior distribution of $\boldsymbol{r}$. This concept of plausibility does not depend on the choice of coordinates, and it does not only reflect the first two moments of the risk factor distribution. Additionally, it can be applied to correlation stress scenarios. The more general notion of plausibility reduces to Maha in the special case of normal distributions with fixed covariance matrix. The relative entropy of the generalised scenario, which is a normal with the same covariance matrix as the prior risk factor distribution, is the Maha between the means of the generalised scenario and of the prior.

(D) The other systematic factors are not fixed but distributed according to the conditional distribution given the values of the fixed risk factors. Denote by $\boldsymbol{r}_D$ the vector of values of the fixed systematic risk factors.

Our first result suggests a choice between these alternatives based on our concept of plausibility. The result says that the specification of partial scenarios as in method (C) or (D) both maximize plausibility. In the literature on stress testing of loan portfolios Bonti et al. [2005] have suggested to use method (D). This is indeed an approach that maximizes plausibility. From our result we learn that we can achieve an equivalent plausibility by using the computationally more efficient approach (C).

We state this result more formally in the following

Proposition 1. *Assume the distribution of systematic risk factors is elliptical with density strictly decreasing as a function of* Maha*. Then:*

1. $\mathrm{Maha}(\boldsymbol{r}_C) = \mathrm{Maha}(\boldsymbol{r}_D)$.
2. *This is the* maximal *plausibility which can be achieved among* all *macro scenarios which agree on the fixed risk factors.*

A proof is in the appendix. This proposition is of high practical relevance. It is the basis of partial scenario analysis. It implies that two choices of macro stress distributions are preferable, namely (C) or (D). Assigning to the non-fixed risk factors other values than the conditional expected values given the fixed risk factors leads to less plausible macro stress scenarios.

2.3 Severe Scenarios

An important disadvantage of stress testing with hand-picked scenarios is the danger to ignore harmful but plausible scenarios. This may create an illusion of safety. A way to overcome this disadvantage is to search systematically for those macro scenarios in some plausible admissibility domain which are most harmful to the portfolio. By searching systematically over admissible domains of plausible macro scenarios one can be sure not to ignore any harmful but plausible scenarios. This is our approach to construct a stress test: find the relevant scenarios which are most harmful yet above some minimal plausibility threshold. This problem can be formulated as an optimization problem which can be solved numerically by using an algorithm of Pistovčák and Breuer [2004].

The admissibility domain is determined by our concept of plausibility. It contains all scenarios with $\mathrm{Maha}(\boldsymbol{r})$ below a threshold k:

$$\mathrm{Ell}_k := \{\boldsymbol{r} : \mathrm{Maha}(\boldsymbol{r}) \leq k\}\,.$$

Geometrically this domain is an ellipsoid whose shape is determined by the covariance matrix of the systematic risk factors:

Partial scenarios do not specify a unique portfolio value but just a distribution, namely the distribution of portfolio values conditional on the values of the risk factors fixed by the scenario. In order to measure the severity of scenarios one needs to quantify the severity of the corresponding conditional portfolio value distribution. In this paper we use the expectation value, although other risk measures could be used as well. Thus we call a partial scenario severe if it has a low conditional expected profit (CEP). To sum up, our stress testing method amounts to solving the following optimization problem

$$\min_{\boldsymbol{r} \in \mathrm{Ell}_k} \mathrm{CEP}(\boldsymbol{r}).$$

The difference between the lowest CEP in the admissibility domain and the CEP in the expected scenario is the maximum expected loss in the admissibility domain. This concept of maximum loss overcomes the problem of dimensional dependence which we mentioned in the introduction. Maximum expected loss over the admissibility domain Ell_k is not affected by excluding or including macro risk factors which are irrelevant to the portfolio value (see Breuer [2008]).

What is the advantage of this worst case search over standard stress testing? First, it achieves a controlled trade-off between plausibility and severity of scenarios. If we want to get more severe scenarios, we choose a higher k and get less plausible worst case scenarios. If we want to get more plausible scenarios, we choose a lower k and get less severe worst case scenarios. Second, it overcomes the historic bias by considering all scenarios which are plausible enough. In this way we can be sure not to miss scenarios which are plausible but did not yet happen in history. Thirdly, worst case scenarios reflect portfolio specific dangers. What is a worst case scenario for one portfolio might be a harmless scenario to another portfolio. This is not taken into account by standard stress testing. Portfolio specific dangers suggest possible counter-action to reduce risk if desired.

2.4 Scenarios Suggesting Risk Reducing Action

Risk reducing action is suggested by identifying the key risk factors which contribute most to the expected loss in the worst case scenario. We define key risk factors as the risk factors with the highest Maximum Loss Constribution (MLC). The loss contribution (LC) of risk factor i to the loss in some scenario $\boldsymbol{r}$ is

$$LC(i, \boldsymbol{r}) := \frac{CEP(\boldsymbol{\mu}) - CEP(\mu_1, \ldots, \mu_{i-1}, r_i, \mu_{i+1}, \ldots \mu_n)}{CEP(\boldsymbol{\mu}) - CEP(\boldsymbol{r})}, \tag{1}$$

if $CEP(\boldsymbol{r}) \neq CEP(\boldsymbol{\mu})$. $LC(i, \boldsymbol{r})$ is the loss if risk factor i takes the value it has in scenario $\boldsymbol{r}$, and the other risk factors take their expected values $\boldsymbol{\mu}$, as a percentage of the loss in scenario $\boldsymbol{r}$. In particular, one can consider the worst case scenario, $\boldsymbol{r} = \boldsymbol{r}^{WC}$. In this case the loss contribution of some risk factor i can be called the Maximum Loss Contribution:

$$MLC(i) := LC(i, \boldsymbol{r}^{WC}). \tag{2}$$

$MLC(i)$ is the loss if risk factor i takes its worst case value and the other risk factors take their expected values, as a percentage of MaxLoss.

The Maximum Loss Contributions of the macro risk factors in general do not add up to 100%. Sometimes the sum is larger, sometimes it is smaller. If this sum is equal to one, the loss in the scenario is exactly equal to the sum of losses from individual risk factor moves. This happens if and only if the risk factors do not interact:

Proposition 2. *Assume CEP as a function of the macro risk factors has continuous second order derivatives. The loss contributions of the risk factors add up to 100% for all scenarios* $\boldsymbol{r}$*,*

$$\sum_{i=1}^{n} LC(i, \boldsymbol{r}) = 1$$

if and only if CEP is of the form

$$CEP(r_1, \ldots, r_n) = \sum_{i=1}^{n} g_i(r_i). \tag{3}$$

This is the case if and only if all cross derivatives of CEP

$$\frac{\partial^2 CEP(\boldsymbol{r})}{\partial r_i \partial r_j} = 0$$

vanish identically for $i \neq j$*.*

This characterization has a substantial practical relevance. The sum of loss contributions measures the amount and direction of risk factor interaction. If the sum is larger than one the interaction between risk factors is benign. The total loss in the scenario is smaller than the sum of losses from individual risk factor moves.

Most dangerous are situations of negative interaction between risk factors. If the sum of single risk factor MLCs is smaller than one the total loss in the scenario is larger than the sum of losses from individual risk factor moves. The harm of the scenario cannot be fully explained by individual risk factor moves. The simultaneous move of some risk factors causes harm on top of the single risk factor moves. In this case it will be necessary to consider Maximum Loss Contributions not of single risk factor moves but of pairs or of larger groups of risk factors. One can Proposition 2 to groups of risk factors.

Consider some partitioning of the risk factor indices $\{1, 2, \ldots, n\}$ into groups $I_1, \ldots, I_s$. Each risk factor will be in exactly one group. The loss contribution of a group I in scenario $\boldsymbol{r}$ can be defined as

$$LC(I, \boldsymbol{r}) := \frac{CEP(a_1, \ldots a_n) - CEP(\boldsymbol{\mu})}{CEP(\boldsymbol{r}) - CEP(\boldsymbol{\mu})}, \tag{4}$$

where $a_i := r_i$ if $i \in I$ and $a_i := \mu_i$ if $i \notin I$. The definition assumes $CEP(\boldsymbol{r}) \neq CEP(\boldsymbol{\mu})$.

Proposition 3. *Assume CEP as a function of the macro risk factors has continuous second order derivatives. The loss contributions of the risk factor groups add up to 100% for all scenarios* $\boldsymbol{r}$*,*

$$\sum_{k=1}^{s} LC(I_k, \boldsymbol{r}) = 1$$

if and only if CEP is of the form

$$CEP(r_1, \ldots, r_n) = \sum_{k=1}^{s} g_k(r_{I_k}),$$

where r_{I_k} denotes the vector containing only the components r_i for $i \in I_k$. This is the case if and only if all cross derivatives of CEP between variables in different groups vanish identically,

$$\frac{\partial^2 CEP(\boldsymbol{r})}{\partial r_i \partial r_j} = 0$$

for each $i \in I_k$ and each $j \in I_l$ with $k \neq l$.

This proposition reveals a weakness in current regulatory thinking. Current regulation analyses portfolio risk along the categories of market and credit risk. Market risk arises from price changes of shares, currencies, interest rates, or commodities. Credit risk describes the harm caused by counterparties not meeting their payment obligations. This is captured in risk factors like PDs, LGDs, default correlations, or other risk factors from which these quantities can be derived. Current regulation determines total risk capital as sum of market and credit risk capital (plus provisions for other risk types). Often it has been argued that total risk capital could be smaller than the sum of market and credit risk because of 'diversification effects'. But this is only justified if the portfolio is separable into one subportfolio just exposed to market risk factors , and another subportfolio just exposed to credit risk factors. Estimates of capital discounts applicable in such situations were made by Rosenberg and Schuermann [2006]. Proposition 3 (for the case of two risk factor groups) implies that a separate calculation of market and credit risk may in fact underestimate the true portfolio risk because it ignores the risks stemming from simultaneous moves in market and credit risk factors. For a detailed discussion of this problem see Breuer et al. [2009].

Key risk factors are essential for the design of possible risk reducing action. One strategy could be to buy hedges which pay off exactly when the key risk factors take their worst case value. Another, more comprehensive but also more expensive strategy is to buy hedges which neutralize the harm done not just by the worst case moves of the key risk factors but by all moves of the key risk factors. For the example of the foreign currency loan portfolios discussed in the next section, this strategy is demonstrated in Breuer et al. [2008].

3 Application: Stress testing a portfolio of foreign currency loans

We now illustrate the concepts and their quantitative significance in an example: a stress test for a portfolio of adjustable rate loans in home or foreign currency (CHF) to 100 borrowers in the rating class B+, corresponding to a default probability of $p_i = 2\%$, or in rating class BBB+, corresponding to a default probability of $p_i = 0.1\%$. At time 0, in order to receive the home currency amount $l =$€ 10 000 the customer of a foreign currency loan takes a loan of $le(0)$ units in a foreign currency, where $e(0)$ is the home currency value of the foreign currency at time 0. The bank borrows $le(0)$ units of the foreign currency on the interbank market. The loan period is one year. During the year the interest rate is fixed every quarter, leading to an average annual interest rate r_h resp. r_f, which is not known in advance. At time 1, which we take to be one year, all the loans expire and the bank repays the foreign currency at the interbank market, and it receives from the customer a home currency amount which is exchanged at the rate $e(1)$ to the foreign currency amount covering repayment of the principal plus interest rolled over from four quarters, plus a spread s_h resp. s_f. So the borrower's payment obligation to the bank at time 1 in home currency is

$$o_f = l\,(1+r_f)\,E + s_f\;l\,E \tag{5}$$

$$o_h = l\,(1+r_h) + s_h\;l \tag{6}$$

for the foreign and home currency loan, where $E := e(0)/e(1)$. The first term on the right hand side is the part of the payment which the bank uses to repay its own loan on the interbank market. The second term is profits remaining with the bank. For all loans in the portfolio we assume they expire at time 1. The model can be extended to a multi period setting allowing for loans not maturing at the same time and requiring payments at intermediate times.

In order to evaluate idiosyncratic and systematic risk of a portfolio of such loans we use a one-period structural model specifying default frequencies and losses given default endogenously. For details of the model we refer to Breuer et al. [2008]. The basic structure of the model is given by the payment obligation distribution derived from the payment obligation function (5) and a log normal payment ability distribution, which involves log normally distributed idiosyncratic changes and an additional dependence of the mean one future GDP changes. (Pesaran et al. [2005a] use a model of this type for the returns of firm value.) Each customer defaults in case their payment ability at the expiry of the loan is smaller than their payment obligation. In case of default the borrower pays what he is able to pay. The difference to the payment obligation first is lost profit and then loss for the bank.

The spread s_h resp. s_f and the variance of the idiosyncratic payment ability changes are determined jointly in a calibration procedure. The first calibration condition ensures that the model default probability coincides with the default probability determined in some external rating procedure. The second calibration condition ensures that expected profit from each loan reaches some target level of €160, which amounts to a return of 20% on a regulatory capital of 8%. Both calibration conditions depend on the spread s and the variance of the idiosyncratic payment ability changes.

The systematic risk factors entering the portfolio valuation are GDP, the home interest rate r_h and the foreign interest rate r_f, and the exchange rate change E. The probability law driving these risk factors is modelled by a time series model that takes account of economic interaction between countries and regions. Estimating the parameters of this model we can simulate scenarios for the systematic risk factors. For details of this model, known in the literature as GVAR model see Pesaran et al. [2001], Pesaran et al. [2005b], Garrett et al. [2006], and Dees et al. [2007].

The profit distribution was calculated in a Monte Carlo simulation by generating 100 000 scenario paths of four steps each. The resulting distribution of risk factors after the last quarter, which is not normal, was used to estimate the covariance matrix of 1yr macro risk factor changes. In each macro scenario defaults of the customers were determined by 100 draws from the idiosyncratic changes in the payment ability distribution. From these we evaluated the profit distribution at the one year time horizon.

3.1 Hand-picked versus systematic stress tests

Let us compare the severity of the hand-picked scenario "GDP shrinks by 3%", which in our model is a 5.42σ event, to the worst case scenario of the same plausibility. Conditional expected profits for the standard scenario 'GDP -3% and other risk factors at their conditional expected value', and of worst case scenarios of the same plausibility, are as follows.

scenario	Maha	CEP
foreign B+		
expected	0	16 001
GDP -3%	5.42	15 950
worst case	5.42	-98 101
foreign BBB+		
expected	0	15 999
GDP -3%	5.42	15 870
worst case	5.42	-95 591
home B+		
expected	0	16 000
GDP -3%	5.42	14 249
worst case	5.42	13 291
home BBB+		
expected	0	16 000
GDP -3%	5.42	15 811
worst case	5.42	15 626

We observe that for all portfolios the conditional expected profits are considerably lower in the worst case scenarios than in the hand-picked GDP scenario. This is evidence of the danger which lies in relying solely on hand-picked scenarios. Expected profits in this rather extreme hand-picked GDP scenario are only slighty lower than unconditional expectations, namely by amounts between €129 and €1 751 on a loan portfolio worth €1m and giving an unconditional expected profit of €16 000. These moderate profit reductions in such an extreme scenario might provide a feeling of safety. But this in an illusion. There are other scenarios out there which are equally plausible but much more harmful: the worst case scenarios like the ones given in Table 1 for different k. Worst case losses are particularly bad for the FX loan portfolios. Instead of expected profits of €16 000 one faces expected losses of €98 101 and €95 591. These huge losses of roughly 11% of loan value are higher than the total regulatory capital of 8%.

3.2 Key risk factors and risk reducing actions

What is a worst case scenario for one portfolio might be a harmless scenario to another portfolio. This is not taken into account by standard stress testing.

Stress testing is relevant only if the choice of scenario takes into account the portfolio structure. In a systematic way this is done by worst case search.

Key risk factors are ones with highest maximum loss contributions (MLC). The worst case scenarios, together with the MLC for each risk factor are given in Table 1, for different sizes of the admissibility domain. These results identify which risk factor is key for which portfolio.

For the foreign currency loan portfolio the exchange rate is clearly the key risk factor. This becomes apparent from Table 1. In the worst case scenario the FX rate alone contributes between 34.4% and 100% of the losses in the worst case scenarios. The MLC of the other risk factors are negligible. This indicates that the FX rate is the key risk factor of the foreign currency loan portfolio.

There is another interesting effect. The dependence of expected profits of foreign currency loans on the CHF/€ rate is not only non-linear, but also not monotone. For the B+ FX loan portfolio (left plot in Figure 1), focusing on changes smaller than 2σ it becomes evident that a small increase in the exchange rate has a positive influence on the portfolio value, but large increases have a very strong negative influence. Correspondingly, in Table 1, if we restrict ourselves to small moves (Maha smaller than 2σ) the worst case scenario is in the direction of increasing exchange rates, but if we allow larger moves the worst case scenario is in the direction of decreasing exchange rates. This effect also shows up in the worst macro scenarios of Table 1. The reason for this non-monotonicity is that a small decrease in the FX rate increases the EUR value of spread payments received. For larger moves of the FX rate this positive effect is outweighed by the increases in defaults due to the increased payment obligations of customers.

The interaction structure of risk factors emerges from the MLC values in Table 1. The first point to observe is that the MLC of single risk factors do not sum up to one. (For example, for the BBB+ FX loan portfolio at $k = 4$ the MLC of single risk factors sum up to 34.4%.) This indicates the presence of dangerous interaction effects between risk factors. The second point is that the pair (IR, FX) has high a MLC. (For example, 85.4% in the case above.) This indicates that interest rate risk considerably enhances exchange rate risk. This is important for the design of risk reducing action. Hedge positions should pay off in situations where exchange rates fall and simultaneously interest rates move up.

The diagnosis that the FX rate is the key risk factor for the foreign currency loans and GDP is the key risk factor for the home currency loans is confirmed by the right and left hand plots in Figure 1, which show the expected profits in dependence of single macro risk factor moves, keeping the other macro risk factors fixed at their expected values. Note the different scales of the two plots. Expected losses of the FX loan are considerably larger than for the home currency loan. This plot also shows that expected profits of both loan types depend non-linearly on the relevant risk factors. The profiles of expected profits in Figure 1 resemble those of short options. A home currency loan behaves like a short put on GDP together with a short call on the home interest rate. A foreign currency loan behaves largely like a short call on the FX rate.

Table 1: Systematic macro stress tests of FX loan portfolios for admissibility domains of different sizes. The top table gives the values of the macro factors along with the CEP in the worst case scenario. For each risk factor the absolute value in the worst case scenario is given along with the risk factor change in standard deviations. The bottom table gives for each worst case scenario the Maximum Loss Contributions of single risk factor changes and of pairs of risk factor changes. IR denotes the interest rate r, FX the exchange rate $e(1)$.

	Worst Macro Scenario						
max.	GDP		IR		FX		
Maha	abs.	stdv	abs.	stdv	abs.	stdv	CEP
FX	**B+**						
2	231.27	-0.10	0.032	0.69	1.420	-1.84	13 180
4	230.95	-0.11	0.039	1.17	1.306	-3.78	-26 138
6	230.85	-0.03	0.046	1.59	1.191	-5.74	-136 009
FX	**BBB+**						
2	231.64	-0.14	0.022	0.03	1.646	2.00	14 855
4	230.92	-0.12	0.039	1.15	1.306	-3.78	-9 931
6	230.81	-0.05	0.046	1.59	1.191	-5.74	-135 202
home	**B+**						
2	228.31	-1.3	0.04	1.07			15 482
4	224.7	-2.64	0.045	2.1			14 458
6	221.04	-4.01	0.049	3.1			12 676
home	**BBB+**						
1	230.06	-0.65	0.038	0.53			15 992
3	226.47	-1.99	0.043	1.57			15 926
5	222.82	-3.35	0.047	2.57			15 710

	Maximum Loss Contribution (MLC)					
max.	single factor moves			moves of pairs		
Maha	GDP	IR	FX	GDP, IR	GDP, FX	IR, FX
FX	**B+**					
2	0.5%	2.6%	51.0%	3.4%	59.1%	89.1%
4	0.0%	0.4%	65.4%	0.5%	70.9%	93.2%
6	0.0%	0.2%	77.0%	0.2%	80.2%	96.7%
FX	**BBB+**					
2	0.0%	0.0%	100.0%	0.0%	100.0%	100.0%
4	0.0%	0.0%	34.4%	0.0%	44.5%	85.4%
6	0.0%	0.0%	75.7%	0.0%	79.3%	96.4%

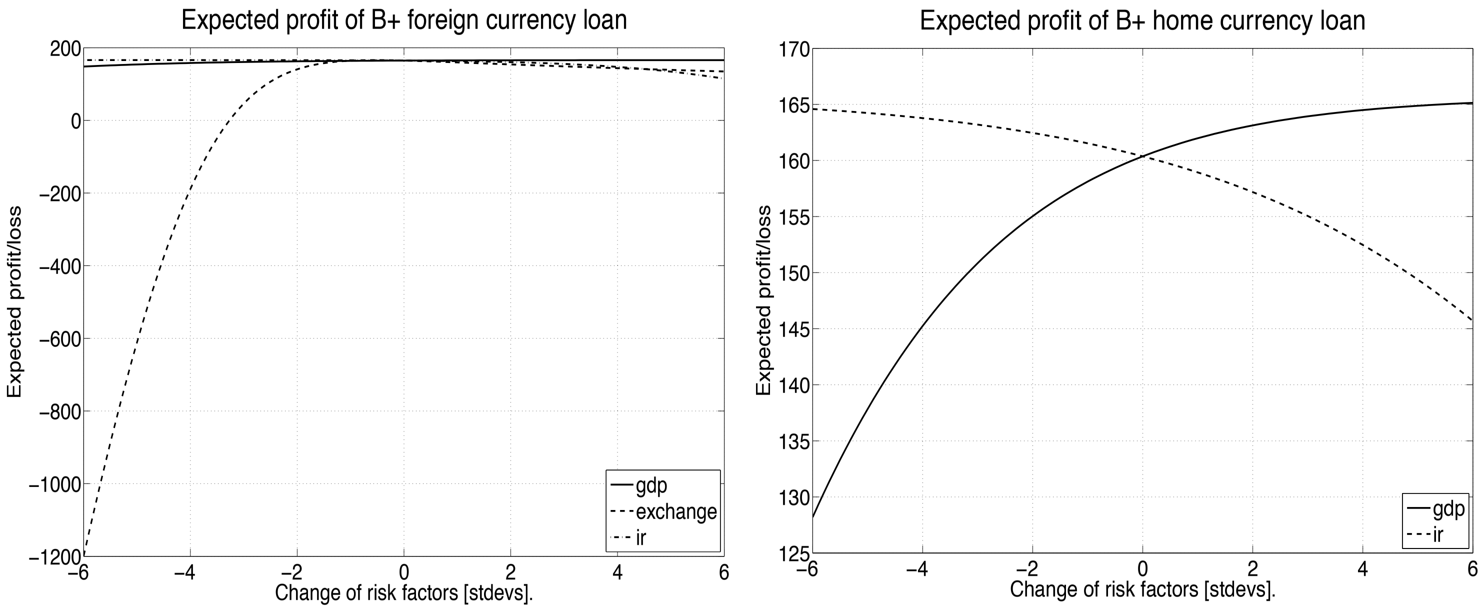

Figure 1: Key risk factors of foreign and home currency loans. Expected profit or loss of a single foreign (left) and home currency (right) loan as a function of changes of the macro risk factors with other macro risk factors fixed at their expected values. The left hand plot shows that for the foreign portfolio the exchange rate is the key risk factor. We also observe the negative effect of small foreign currency depreciations, which is particularly pronounced for the BBB+ portfolio. The right hand plot shows that for the home portfolio GDP is the key risk factor. Note the different scales of the two plots.

One could ask why the effort to search for worst case scenarios is necessary to identify key risk factors. Wouldn't it be easier to read the key risk factors from the plots in Figure 1? This would be true if losses from moves in different risk factors *added up*. But for certain kinds of portfolios the worst case is a *simultaneous* move of several risk factors—and the loss in this worst case might be considerably worse than adding up the losses resulting from moves in single risk factors. This is the message of Proposition 2. The effects of simultaneous moves are not reflected in Figure 1, but they do show up in the worst case scenarios.

The identification of key risk factors suggests risk reducing counter-actions. Knowing that the exchange rate is the key risk factor for FX loans, one can plot the behaviour of CEP in dependence of exchange rate moves, as in the left hand plots of Fig 1. Breuer et al. [2008] show how FX derivatives can be used to construct hedges reducing the exchange rate risk of foreign currency loans. It turns out that FX options can be used to virtually eliminate the dependence of expected loss on exchange rates—at some fixed level of interest rates and other macroeconomic factors. But the hedge is not perfect: Firstly it cannot fully remove dependence of expected losses on exchange rates at other levels of interest rates, and secondly it can bring to zero only the expectation but not the variance of losses caused by adverse exchange rate moves.

4 Conclusion

The central message of our paper is that the three principles of the Basel Committee on Banking Supervision [2005] required for stress tests, *Plausibility*, *Severity* of stress scenarios and *Suggestiveness of risk reducing action*, can be systematically implemented within a standard quantitative risk management framework. In order to do so we need a measure of plausibility that can be formulated using the probability distribution of the risk factors but that does not suffer from the dimensional dependence of maximum loss. We show that this concept of plausibility can be formulated by working with regions of a given Mahalanobis radius instead than working with regions of given probability mass. We need to replace the common practice of hand picked scenarios by a systematic worst case search over the given region of plausibility. Finally we have to identify the key risk factors and their contributions to maximum loss. The key contribution to maximum loss may be only revealed if we take into account simultaneous moves in riks factors.

Our approach has three major advantages compared to standard stress tests First, it ensures that no harmful scenarios are missed and therefore prevents a false sense of safety. Second, it does not analyse scenarios which are too implausible and would therefore jeopardize the credibility of stress analysis. Third, it allows for a portfolio specific identification of key risk factors. We hope that the compatibility of our concepts with the standard quantitative risk management framework used by practcioners make the insights of this paper useful in practcial stress testing problems.

References

Basel Committee on Banking Supervision. International convergence of capital measurement and capital standards. a revised framework. Technical report, Bank for International Settlements, 2005.

Gabriel Bonti, Michael Kalkbrener, Christopher Lotz, and Gerhard Stahl. Credit risk concentration under stress. In *Concentration Risk in Credit Portfolios*. Deutsche Bundesbank, 2005. Available at `http://www.bis.org/bcbs/events/crcp05bonti.pdf`.

Thomas Breuer. Overcoming dimensional dependence of worst case scenarios and maximum loss. *Journal of Risk*, 11(1):79–92, 2008.

Thomas Breuer and Gerald Krenn. Stress testing. Guidelines on Market Risk 5, Oesterreichische Nationalbank, Vienna, 1999. Also available as `http://www.oenb.at/en/img/band5ev40_tcm16-20475.pdf`.

Thomas Breuer, Martin Jandačka, Klaus Rheinberger, and Martin Summer. Hedge the stress. Using stress tests to design hedges for foreign currency loans. In Daniel Rösch and Harald Scheule, editors, *Stress Testing for Financial Institutions—Applications, Regulations, and Techniques*, pages 111–126. Risk Books, 2008.

Thomas Breuer, Martin Jandačka, Klaus Rheinberger, and Martin Summer. Does adding up of economic capital for market- and credit risk amount to conservative risk assessment? *Journal of Banking and Finance*, forthcoming, 2009.

Stephane Dees, Filippo di Mauro, Hashem M. Pesaran, and L. Vanessa Smith. Exploring the international linkages of the Euro area: A global VAR analysis. *Journal of Applied Econometrics*, 22(1), 2007.

Kai-Tai Fang, Samuel Kotz, and Kai-Wang Ng. *Symmetric Multivariate and Related Distributions*, volume 36 of *Monographs in Statistics and Probability*. Chapman and Hall, London, 1987.

Anthony Garrett, Hashem M. Pesaran, and Yongcheal Shin. *Global and National Macroeconomic Modelling*. Oxford University Press, 2006.

Paul H. Kupiec. Stress testing in a value at risk framework. *Journal of Derivatives*, 6:7—-24, 1998.

Hashem M. Pesaran, Til Schuermann, and Scott M. Weiner. Modelling regional interdependencies using a global error-correcting macroeconometric model. URL `http://ideas.repec.org/p/cam/camdae/0119.html`. Cambridge Working Papers in Economics 0119, Faculty of Economics (formerly DAE), University of Cambridge, 2001.

Hashem M. Pesaran, Til Schuermann, and Björn-Jakob Treutler. Global business cycles and credit risk. Technical Report NBER Working Paper No. W11493, NBER, 2005a. Available at SSRN: `http://ssrn.com/abstract=762771`.

Hashem M. Pesaran, L. Vanessa Smith, and Ron P. Smith. What if the UK had joined the Euro in 1999? An empirical evaluation using a global VAR. URL `http://ideas.repec.org/p/scp/wpaper/05-24.html`. Institute of Economic Policy Research Working Papers 05.24, 2005b.

Filip Pistovčák and Thomas Breuer. Using Quasi Monte Carlo-Scenarios in risk management. In Harald Niederreiter, editor, *Monte Carlo and Quasi-Monte Carlo-Methods 2002*, pages 379–392. Springer, 2004.

Joshua V. Rosenberg and Til Schuermann. A general approach to integrated risk management with skewed, fat-tailed risks. *Journal of Financial Economics*, 79(3):569–614, 2006. Previous version as Federal Reserve Bank of New York Staff Report 185 (2004) at `http://www.newyorkfed.org/research/staff_reports/sr185.html`.

Gerold Studer. *Maximum Loss for Measurement of Market Risk*. Dissertation, ETH Zürich, Zürich, 1997. Also available as `http://www.gloriamundi.org/picsresources/gsmlm.pdf`.

Gerold Studer. Market risk computation for nonlinear portfolios. *Journal of Risk*, 1(4):33–53, 1999.

Martin Čihák. Stress testing: A review of key concepts. CNB Internal Research and Policy Note 2, Czech National Bank, 2004. `www.cnb.cz/en/pdf/IRPN_2_2004.pdf`.

Martin Čihák. Introduction to applied stress testing. Technical Report WP/07/59, IMF, 2007. `http://www.imf.org/external/pubs/ft/wp/2007/wp0759.pdf`.

5 Proof of Proposition 1

Let us assume that we have n macro risk factors, whose change is governed by a multivariate elliptically symmetric distribution with covariance matrix Cov and mean $\boldsymbol{\mu}$. Let us assume that the risk factors are indexed in such a way that the fixed risk factors have numbers $1, 2, \ldots, k$. Let us denote by $r^*_{k+1}, \ldots, r^*_n$ the conditional expected values of risk factors $r_{k+1}, \ldots, r_n$ given that $r_1, \ldots, r_k$ have their fixed values. Breuer [2008] shows that $\mathrm{Maha}(r_1, \ldots r_k, r^*_{k+1}, \ldots, r^*_n) = \mathrm{Maha}(r_1, \ldots r_k, r^*_{k+1}, \ldots, r^*_{n-1})$ and that choosing $r_n = r^*_n$ minimises Maha among all scenarios with the values of the first $n-1$ risk factors equal to $r_1, \ldots r_k$, $r^*_{k+1}, \ldots, r^*_{n-1}$. Repeating this argument for the risk factors r_{n-1} down to r_{k+1} yields the Proposition.

6 Proof of Proposition 2

Let f be a real-valued function with continuous second order derivatives on some domain in $\mathbb{R}^n$. Consider two points $X^0, X^1 \in \mathbb{R}^n$ such that the cube with diagonal points X^0, X^1 is in the domain of definition of f. f plays the role of the objective function CEP, X^0 is the expected scenario $\boldsymbol{\mu}$, and X^1 is some arbitrary scenario $\boldsymbol{r}$.

We use the following short hand notation. For a vector $i = (i_1, \ldots, i_n)$ of ones and zeros write

$$f(i_1...i_n) := f(x_1^{i_1}, x_2^{i_2}, ..., x_n^{i_n}).$$

For an index vector i with only component $i_j = 1$ and all other components equal to zero we write $f_j := f(i_1...i_n)$. For an index vector i with only the two components $i_j = i_k = 1$ and all other components equal to zero we write $f_{jk} := f(i_1...i_n)$.

Lemma 1. *If f has continuous second order derivatives on the cube with diagonal points X^0, X^1 the value of the function f in scenario X^1 equals*

$$f(1...1) = \sum_{i=1}^{n} f_i - (n-1)f(0...0) + \sum_{1 \le i < j \le n} I_{ij}, \tag{7}$$

where

$$I_{ij} = \int_{x_i^0}^{x_i^1} \int_{x_j^0}^{x_j^1} \frac{\partial^2 f}{\partial x_i \partial x_j}(x_1^0, x_2^0, ..., u_i, x_{i+1}^0, ..., u_j, x_{j+1}^1, ..., x_n^1) du_j du_i,$$

for $1 \le i < j \le n$.

Proof. We proceed by induction in the number of variables, n. For $n = 1$, f is a function of one variable and eq. (7) reduces to $f(1) = f(1)$. The inductive step will use also eq. (7) for functions of $n = 2$ variables, so we prove it separately. For $n = 2$ we get

$$\begin{aligned} f(x_1^1, x_2^1) &= f(x_1^0, x_2^1) + \int_{x_1^0}^{x_1^1} \frac{\partial f}{\partial x_1}(u_1, x_2^1) du_1 \\ &= f_2 + \int_{x_1^0}^{x_1^1} \left(\frac{\partial f}{\partial x_1}(u_1, x_2^0) + \int_{x_2^0}^{x_2^1} \frac{\partial f}{\partial x_1 x_2}(u_1, u_2) du_2 \right) du_1 \\ &= f_2 + f_1 - f(00) + I_{12}, \end{aligned} \tag{8}$$

which proves eq. (7) for functions of $n = 2$ variables.

Now we assume that eq. (7) holds for functions of n variables and show that it holds for a funtion f of $n + 1$ variables. Define the function

$$h(x_1, ..., x_n) := f(x_1, ..., x_n, x_{n+1}^1).$$

Eq. (7) for h reads

$$f(1...11) = \sum_{i=1}^{n} f_{i(n+1)} - (n-1)f(0...01) + \sum_{1 \le i < j \le n} I_{ij}. \tag{9}$$

For the function $g_{i(n+1)}$ of two variables defined by

$$g_{i(n+1)}(x_i^1, x_{n+1}^1) := f_{i(n+1)}.$$

eq. (8) reads

$$f_{i(n+1)} = f_i + f_{n+1} - f(0...0) + I_{i(n+1)}. \tag{10}$$

Substituting eq. (10) into the eq. (9) we get

$$\begin{aligned} f(1...1) &= \sum_{i=1}^{n} \left(f_i + f_{n+1} - f(0...0) + I_{i(n+1)}\right) \\ &\quad -(n-1)f_{n+1} + \sum_{1\le i<j\le n} I_{ij} \\ &= \sum_{i=1}^{n+1} f_i - nf(0...0) + \sum_{1\le i<j\le n+1} I_{ij}, \end{aligned} \tag{11}$$

which is eq. (7) for the function f of $n+1$ variables. This finishes the proof of Lemma 1. $\square$

Lemma 1 gives a simple approximation of the change of f between two points X^0, X^1:

$$f(1...1) - f(0...0) \approx \sum_{i=1}^{n} \left(f_i - f(0...0)\right). \tag{12}$$

The approximation error is

$$\epsilon = \sum_{1\le i<j\le n} I_{ij}. \tag{13}$$

If the function f represents portfolio values, the left side of eq. (12) represents portfolio profit changes when moving from scenario X^0 to scenario X^1. The right side is the sum of contributions of the individual risk factors. The error term ϵ describes the interaction between the risk factors. It is bounded by

$$|\epsilon| \le K\frac{(n-1)n}{2}||X_n^1 - X_n^0|| \tag{14}$$

if the absolute values of second order mixed derivatives are bounded by some constant K. As a consequence, the approximation (12) is exact when the second ordered mixed derivatives vanish everywhere in the cube with diagonal points X^0, X^1. The following Lemma also establishes the converse.

Lemma 2. *Assume f has continuous second order derivatives. The following two statements are equivalent:*
(1) For any two X^0, X^1

$$f(1...1) - f(0...0) = \sum_{i=1}^{n} \left(f_i - f(0...0)\right) \tag{15}$$

(2) for all X and for all i, j, $1 \le i < j \le n$

$$\frac{\partial^2 f}{\partial x_i \partial x_j}(X) = 0 \tag{16}$$

Proof. Eq. (16) implies eq. (15) directly by the definition of I_{ij} and Lemma 1: If for all X in the cube with diagonal points X^0, X^1 and for all pairs (i,j) we have $\partial^2 f/\partial x_i \partial x_j(X) = 0$, then by definition of I_{ij} we have $I_{ij} = 0$. Eq. (15) is implied by eq. (7).

Eq. (15) implies eq. (16) as follows. Assume there is some X and some i, j for which $\partial^2 f/\partial x_i \partial x_j(X) > 0$ (resp. < 0). Then the continuity of the second order derivatives implies the existence of a neighbourhood $O(X)$ contained in the cube, such that $\partial^2 f/\partial x_i \partial x_j(Y) > 0$ (resp. < 0) for all Y in $O(X)$. Take two scenarios X^0, X^1 in $O(X)$ such, that $x_k^0 < x_k^1$ for $k \in \{i,j\}$ and $x_k^0 = x_k^1$ for $k \notin \{i,j\}$. Then from the definition of I_{ij} we get $I_{ij} > 0$ (resp. < 0) and $I_{kl} = 0$ for $(k,l) \neq (i,j)$. From eq. (7), we get $f(1...1) - f(0...0) > \sum_{i=1}^n (f_i - f(0...0))$ if $\partial^2 f/\partial x_i \partial x_j > 0$, resp. $f(1...1) - f(0...0) < \sum_{i=1}^n (f_i - f(0...0))$ if $\partial^2 f/\partial x_i \partial x_j < 0$. This finishes the proof of Lemma 2. □

Lemma 3. *Assume f has continuous second order derivatives. The following two statements are equivalent:*
(1) For all X and for all i, j, $1 \leq i < j \leq n$

$$\frac{\partial^2 f}{\partial x_i \partial x_j}(X) = 0 \tag{16}$$

(2) f can be written as

$$f(x_1, x_2, ..., x_n) = \sum_{i=1}^{n} g_i(x_i). \tag{17}$$

Proof. (17) implies (16) by direct derivation. (16) implies (17) by induction in the number of variables, n. For $n = 2$ assume that all cross derivatives vanish. Then $\partial f(X)/\partial x_1 = h(x_1)$, resp. $f(X) = H(x_1) + g_2(x_2)$. Choosing $g_1(x_1) = H(x_1)$ we get eq. (17) for $n = 2$:

$$f(X) = \sum_{i=1}^{2} g_i(x_i). \tag{18}$$

In the induction step assume that (16) implies (17) for functions of n variables. Take a function f of $n+1$ variables with continuous second order derivatives. First we will show that f can be written as

$$f(X) = u_j(x_1, x_j, ..., x_{n+1}) + v_j(x_2, ..., x_{n+1}), \tag{19}$$

for $j = 2, ..., n+1$. For $j = 2$ we can take $u_2 = f$ and $v_2 = 0$.
Now assume that the separation is possible up to component j. As the function v_j does not depend on the variable x_1 we get

$$\frac{\partial^2 f}{\partial x_1 \partial x_j} = \frac{\partial^2 u_j}{\partial x_1 \partial x_j}$$

which equals zero because of the induction basis (16). Applying (18) to $u_j(x_1, x_j, ..., x_{n+1})$, regarded as a function of x_1 and x_j, we get

$$u_j(x_1, x_j, ..., x_{n+1}) = u_{j+1}(x_1, x_{j+1}, ..., x_{n+1}) + h_j(x_j, ..., x_{n+1}).$$

Denoting $v_{j+1} := v_j + h$ we get

$$f(X) = u_{j+1}(x_1, x_{j+1}, ..., x_{n+1}) + v_{j+1}(x_2, ..., x_{n+1}).$$

For $j = n+1$ (19) gives

$$f(X) = u_{n+1}(x_1, x_{n+1}) + v_{n+1}(x_2, ..., x_{n+1}). \tag{20}$$

As v_{n+1} does not depend on x_1, we infer from eq. (16)

$$\frac{\partial^2 u_{n+1}}{\partial x_1 \partial x_{n+1}} = \frac{\partial^2 f}{\partial x_1 \partial x_{n+1}} = 0.$$

Applying again eq. (18) to the function u_{n+1} we get

$$u_{n+1}(x_1, x_{n+1}) = g_1(x_1) + h(x_{n+1}).$$

and thus

$$f(X) = g_1(x_1) + h(x_{n+1}) v_{n+1}(x_2, ..., x_{n+1}).$$

For $i, j \in \{2, \ldots, n+1\}$ we get from (16)

$$\begin{aligned} 0 &= \frac{\partial^2 f}{\partial x_i \partial x_j}(X) = \frac{\partial^2 g_1(x_1)}{\partial x_i \partial x_j} + \frac{\partial^2}{\partial x_i \partial x_j}\left(h(x_{n+1}) + v_{n+1}(x_2, ..., x_{n+1})\right) \\ &= \frac{\partial^2 v_{n+1}}{\partial x_i \partial x_j}(x_2, ..., x_{n+1}) \end{aligned}$$

The function v_{n+1} is a function of n variables with all mixed second orders derivatives equal to zero. From the assumption of the induction step we get (17) for the function f of $n+1$ variables. $\square$

The defintion of the Loss Contribution in eq.(1) can be written as

$$LC(i, \boldsymbol{r}) := \frac{f_i - f(0...0)}{f(1...1) - f(0...0)}, \tag{21}$$

assuming $f(0...0) \neq f(1...1)$, and taking $X^0 = \boldsymbol{\mu}, X^1 = \boldsymbol{r}, f = CEP$. Lemmata 2 and 3 imply that

$$\sum_{i=1}^{n} LC(i, \boldsymbol{r}) = 1$$

holds for all $\boldsymbol{r}$ if and only if the function CEP can written as a sum (17) resp. if and only if the second order derivatives vanish identically.

7 Proof of Proposition 3

Proof. Let f be a real-valued function with continuous second order derivatives on some domain in $\mathbb{R}^n$. Consider two points $X^0, X^1 \in \mathbb{R}^n$ such that the cube with diagonal points X^0, X^1 is in the domain of definition of f. f plays the role of the objective function CEP, X^0 is the expected scenario $\boldsymbol{\mu}$, and X^1 is an arbitrary scenario $\boldsymbol{r}$. In addition to the notation $f(i_1 i_2 \ldots i_n)$ and f_i introduced in the proof of Proposition 2 we use

$$f_I \;:=\; f(i_1 ... i_n),\ i_k = \left\{ \begin{array}{ll} 1 & k \in I \\ 0 & k \notin I \end{array} \right. ,$$

We need the following Lemma.

Lemma 4. *Assume f has continuous second order derivatives. Then for any two scenarios $X^0, X^1 \in \mathbb{R}^n$*

$$f(1\dots1) - f(0\dots0) = \sum_{k=1}^{s} (f_{I_k} - f(0\dots0)) + \sum_{1\le k<l\le s} \tilde{I}_{kl}, \tag{22}$$

where

$$\tilde{I}_{kl} := \int_0^1 \int_0^1 \sum_{k\in I_k, l\in I_l} (x_k^1 - x_k^0)(x_l^1 - x_l^0) \frac{\partial^2 f^n}{\partial x_k \partial x_l}(y_1, ..., y_n) du dv$$

with

$$y_i = \begin{cases} x_i^0 + u(x_i^1 - x_i^0) & i \in I_k, \\ x_i^0 + v(x_i^1 + x_i^0) & i \in I_l, \\ x_i^1 & i > \max(I_l) \\ x_i^0 & \text{otherwise} \end{cases} \tag{23}$$

Proof. Define a function $\tilde{f} : \mathbb{R}^s \to \mathbb{R}$ as

$$\tilde{f}(y_1, \dots, y_s) := f(g_1(y_1), ..., g_s(y_s)), \tag{24}$$

with

$$g_k(t) := X_{I_k}^0 + t\left(X_{I_k}^1 - X_{I_k}^0\right) \tag{25}$$

for $0 \le t \le 1$ and $0 \le k \le s$. Applying Lemma 1 to $\tilde{f}$ we get eq. (22) and $I_{kl} = \tilde{\tilde{I}}_{kl}$. □

Proposition 3 follows from applying Lemmata 2 and 3 to $\tilde{f}$. □

Index of Working Papers:

June 19, 2006	David Laidler	128	Three Lectures on Monetary Theory and Policy: Speaking Notes and Background Papers
July 9, 2006	Ansgar Belke, Bernhard Herz, Lukas Vogel	129	Are Monetary Rules and Reforms Complements or Substitutes? A Panel Analysis for the World versus OECD Countries
August 31, 2006	John Williamson (comment by Marc Flandreau)	130	A Worldwide System of Reference Rates
September 15, 2006	Sylvia Kaufmann, Peter Kugler	131	Expected Money Growth, Markov Trends and the Instability of Money Demand in the Euro Area
September 18, 2006	Martin Schneider, Markus Leibrecht	132	AQM-06: The Macroeconomic Model of the OeNB
November 6, 2006	Erwin Jericha and Martin Schürz	133	A Deliberative Independent Central Bank
December 22, 2006	Balázs Égert	134	Central Bank Interventions, Communication and Interest Rate Policy in Emerging European Economies
May 8, 2007	Harald Badinger	135	Has the EU's Single Market Programme fostered competition? Testing for a decrease in markup ratios in EU industries
May 10, 2007	Gert Peersman	136	The Relative Importance of Symmetric and Asymmetric Shocks: the Case of United Kingdom and Euro Area
May 14, 2007	Gerhard Fenz and Martin Schneider	137	Transmission of business cycle shocks between unequal neighbours: Germany and Austria
July 5, 2007	Balázs Égert	138	Real Convergence, Price Level Convergence and Inflation Differentials in Europe
January 29, 2008	Michał Brzoza-Brzezina, Jesus Crespo Cuaresma	139	Mr. Wicksell and the global economy: What drives real interest rates?

March 6, 2008	Helmut Stix	140	Euroization: What Factors drive its Persistence? Household Data Evidence for Croatia, Slovenia and Slovakia
April 28, 2008	Kerstin Gerling	141	The Real Consequences of Financial Market Integration when Countries Are Heterogeneous
April 29, 2008	Aleksandra Riedl and Silvia Rocha-Akis	142	Testing the tax competition theory: How elastic are national tax bases in Western Europe?
May 15, 2008	Christian Wagner	143	Risk-Premia, Carry-Trade Dynamics, and Speculative Efficiency of Currency Markets
June 19, 2008	Sylvia Kaufmann	144	Dating and forecasting turning points by Bayesian clustering with dynamic structure: A suggestion with an application to Austrian data.
July 21, 2008	Martin Schneider and Gerhard Fenz	145	Transmission of business cycle shocks between the US and the euro area
September 1, 2008	Markus Knell	146	The Optimal Mix Between Funded and Unfunded Pensions Systems When People Care About Relative Consumption
September 8, 2008	Cecilia García-Peñalosa	147	Inequality and growth: Goal conflict or necessary prerequisite?
September 30, 2008	Fabio Rumler and Maria Teresa Valderrama	148	Comparing the New Keynesian Phillips Curve with Time Series Models to Forecast Inflation
January 30, 2009	Claudia Kwapil, Johann Scharler	149	Expected Monetary Policy and the Dynamics of Bank Lending Rates
February 5, 2009	Thomas Breuer, Martin Jandačka, Klaus Rheinberger, Martin Summer	150	How to find plausible, severe, and useful stress scenarios